THE DISCRETE GAUSS IMAGE PROBLEM

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Abstract. The Gauss Image problem is a generalization to the question originally posed by Aleksandrov, who studied the existence of the convex body with prescribed Aleksandrov's integral curvature. A simple discrete case of the Gauss Image Problem can be formulated as follows: given a finite set of directions in $\mathbb{R}^n$ and the same number of unit vectors, does there exist a convex polytope in $\mathbb{R}^n$ containing the origin in its interior with vertices at given directions such that each normal cone at the vertex contains exactly one of the given vectors in its interior?

We pose a combinatorial problem, called the Assignment Problem, for discrete measures. It is shown that the discrete Gauss Image Problem and the Assignment Problem are equivalent. We establish a geometric condition for measures which solves the Assignment Problem and, hence, the Gauss Image Problem. We also show that generically, the Assignment Problem has a solution. The proper reformulation for the uniqueness questions is also addressed and analyzed. The work establishes interesting connections of the Discrete Gauss Image problem to Hall's marriage theorem and transportation polytopes.

Contents

Date: October 30, 2022.

2010 *Mathematics Subject Classification.* 52A20, 52A38, 52A40, 52B11, 35J20, 35J96.

Key words and phrases. Convex Geometry, The Gauss Image Problem, Aleksandrov Condition, Monge-Ampère equation, Aleksandrov Problem.

1. Introduction

The Brunn-Minkowski theory, originating back to Brunn in the 19th century, is the core part of the study of geometry of convex bodies. An essential part of it is the study of Minkowski problems, characterizing measures associated with convex bodies. These problems have a huge influence on Brunn-Minkowski theory and mathematics outside of convexity, such as the subjects of fully non-linear partial differential equations, optimal mass transport, and others.

More recently, two extensions of Brunn-Minkowski theory were considered. The L_p Brunn-Minkowski theory was initially proposed by Firey in [17] but seriously investigated only later by Lutwak [25]. And the dual Brunn-Minkowski theory, which originated with Lutwak [26], on dual mixed volumes. Similarly, many analogues of Minkowski problems were established in the L_p Brunn-Minkowski theory and the dual Brunn-Minkowski Theory. The study of these problems are an essential part of the development of these previously mentioned subjects. Furthermore, they have lead to some non-trivial and important conjectures, such as, for example, the log-Brunn-Minkowski inequality (see [6, 13, 20, 24, 42]). We refer the reader to Chapters 8 and 9 of Schneider's textbook [43] and to articles [8, 11, 12, 21–23, 25, 27–31, 34–36, 44, 48–52] for an overview of Minkowski problems. And refer to [5, 7, 33, 37, 45] for information on the regularity of the solutions.

The Gauss Image problem, introduced in [14], is an analogue of the Minkowski problem. The objective is to characterize push forward measures by the multi-valued radial Gauss map, also known as the radial Gauss Image. Many important measures, such as the integral curvature defined by Aleksandrov [3], surface are measures of Aleksandrov-Fenchel-Jessen [2] and, more recently, the dual curvature measures [22], arise through this push forward. This motivates the characterization of push forward measures by the radial Gauss map; that is, the Gauss Image Problem. Since dual curvature measures are pushforwards by the Gauss map, this problem can be seen as vital element of connecting Brunn-Minkowski theory with dual Brunn-Minkowski theory [22].

The radial Gauss image map is a composition of the multivalued Gauss map and the radial map. More precisely, given $K \in \mathcal{K}_o^n$ (where $\mathcal{K}_o^n$ is the set of convex bodies containing the origin in their interiors), we define the radial Gauss image of $\omega \subset S^{n-1}$ as:

$$\boldsymbol{\alpha}_K(\omega) = \bigcup_{x \in r_K(\omega)} N(K, x) \subset S^{n-1}. \tag{1.1}$$

Here $r_K : S^{n-1} \to \partial K$ is the radial map of K, which is defined for $u \in S^{n-1}$ by $r_K(u) = ru \in \partial K$, with positive r and $N(K, x)$ is the set of all outer unit normals to a boundary point of K, defined below:

$$N(K, x) = \{v \in S^{n-1} : (y - x) \cdot v \leq 0 \text{ for all } y \in K\}. \tag{1.2}$$

Definition 1.1. Suppose λ is a submeasure on S^{n-1} defined on a spherical Lebesgue measurable sets and $K \in \mathcal{K}_o^n$. Then $\lambda(K, \cdot)$, *the Gauss image measure of* λ *via* K, is a submeasure defined as the pushforward of the λ via map $\boldsymbol{\alpha}_K$. That is for each borel $\omega \subset S^{n-1}$

$$\lambda(\boldsymbol{\alpha}_K(\omega)) = \lambda(K, \omega) \tag{1.3}$$

Given two spherical Borel measures λ and μ, *the Gauss Image Problem* asks what are necessary and sufficient conditions for a body $K \in \mathcal{K}_o^n$ to satisfy $\mu = \lambda(K, \cdot)$. If the solution exists, we are also interested in the question of uniqueness. When λ is spherical Lebesgue measure, $\lambda(K,\cdot)$ is known as Aleksandrov's integral curvature of the body K [3]. When λ is Federer's $(n-1)^{\text{th}}$ curvature measure, $\lambda(K,\cdot)$ is the surface area measure of Alekxandrov-Fenchel-Jessen [2]. Finally, more the more recently defined dual curvature in dual Brunn-Minkowski theory [22] is known as the Gauss image measure.

Special cases of the Gauss Image Problem were investigated by numerous people. When λ is spherical Lebesgue measure, it was studied by Aleksandrov in [1] and [3]. Different proofs were given by Oliker [36] and Bertand [9]. When one of the measures is assumed to be absolutely continuous, the question was studied in [14] by Böröczky, Lutwak, Yang, Zhang and Zhao (which implies previous results). In this case, $\lambda(K,\cdot)$ is always a Borel measure. There, the Aleksandrov relation was introduced to attack the problem:

Definition 1.2. Two Borel measures μ and λ on S^{n-1} are called *Aleksandrov related* if

$$\lambda(S^{n-1}) = \mu(S^{n-1}) > \mu(\omega) + \lambda(\omega^*), \tag{1.4}$$

for each compact, spherically convex (See Section 2) set $\omega \subset S^{n-1}$. The set ω^* is defined as a polar set:

$$\omega^* := \bigcap_{u\in\omega} \{v \in S^{n-1} : uv \le 0\} \tag{1.5}$$

The following solution to the Gauss Image problem was obtained:

Theorem 1.3 (K. J. Böröczky, E. Lutwak, D. Yang, G. Y. Zhang and Y. M. Zhao [14])**.** *Suppose μ and λ are Borel measures on S^{n-1} and λ is absolutely continuous. If μ and λ are Aleksandrov related, then there exists a $K \in \mathcal{K}^n_o$ such that $\mu = \lambda(K,\cdot)$.*

Moreover, it was shown that the Aleksandrov relation is a necessary assumption for the existence of a solution to the Gauss Image problem, if one of the measures is assumed to be absolutely continuous and strictly positive on open sets [14]. In this case, the solution to the Gauss Image problem is shown to be unique up to a dilation. The L_p analogues of the Aleksandrov problem were considered by Huang, Lutwak, Yang and Zhang in [21], by Mui in [32], and by Zhao in [50]. The L^p analogue of the Gauss Image Problem was considered in [46] by C. Wu, D. Wu, and Xiang.

In this work, we would like to address the discrete direction of the Gauss Image Problem. In the most simple form, it can be formulated as follows:

In $\mathbb{R}^n$, suppose we are given two sets of unit vectors $\{v_1 \dots v_m\}$ and $\{u_1 \dots u_m\}$. Suppose $\{v_1 \dots v_m\} \subset S^{n-1}$ are not contained in any closed hemisphere. Let $\mathcal{P}$ be the set of convex polytopes containing the origin in their interiors with vertices at the v_i directions. We ask the following question: what are necessary and sufficient conditions on vectors v_i, u_j for the existence of a convex polytope $P \in \mathcal{P}$, such that every normal cone at each vertex of P contains exactly one vector from the set $\{u_1 \dots u_m\}$ in its interior, and each vector from the set $\{u_1 \dots u_m\}$ is contained in exactly one normal cone?

Considering the dual problem, this simply stated question can be seen to be very similar in spirit to the famous question posed by Minkowski for polytopes: the Minkowski problem asks whether there exist a polytope with specified directions and measures of the facets. We ask about the existence of a polytope with specified directions of the facets, such that each facet is penetrated by exactly one out of specified vectors in its interior.

Given discrete measures μ and λ:

$$\lambda = \sum_{j=1}^{k} \lambda_j \delta_{u_j}$$
$$\mu = \sum_{i=1}^{m} \mu_i \delta_{v_i}. \tag{1.6}$$

We introduce *the Discrete Gauss Image problem*, which asks whether there exists a body $K \in \mathcal{K}_o^n$ such that $\mu = \lambda(K, \cdot)$. If such a K exists, we can always find a polytope $P \in \mathcal{K}_o^n$ with vertices $r_K(v_i)$, that is vertices at radial directions v_i, such that $\mu = \lambda(K, \cdot) = \lambda(P, \cdot)$ (see Proposition 4.1). Thus, it suffices to restrict the search for the polytope solution with vertices at v_i directions.

The problem, in its most generality, has a natural algebraic obstacle. Firstly, note that if a polytope P with vertices $r_P(v_i)$ is a solution to the Discrete Gauss Image Problem, then the u_j's are contained in the interiors of normal cones of vertices. Otherwise, if some u_j is contained on the boundary of normal cone, $\lambda(P, \cdot)$ would not be a finite measure. Define a function f from $\{1 \dots k\} \to \{1 \dots m\}$ such that $f_P(j) = i$, if and only if $u_j \in \mathring{\boldsymbol{\alpha}}_P(v_i)$. Then if P is a solution, $\mu(\{v_i\}) = \lambda(P, \{v_i\})$, and we obtain for each i:

$$\sum_{j \in f_P^{-1}(i)} \lambda_j = \mu_i. \tag{1.7}$$

Given two discrete measures, such function might not exist in principle. Therefore, each Discrete Gauss Image problem comes with the combinatorial problem of finding an f assigning vectors u_j to normal cones v_i, satisfying the condition above on the weights. This motivates the next definition.

Definition 1.4. Given two discrete measures λ and μ, we associate *the set of assignment functions*:

$$\mathbb{F}_{\mu,\lambda} := \Big\{ f\colon \{1 \dots k\} \to \{1 \dots m\} \mid \sum_{j \in f^{-1}(i)} \lambda_j = \mu_i \Big\}. \tag{1.8}$$

We call $f \in \mathbb{F}$ *an assignment function* with respect to μ and λ. If for $f \in \mathbb{F}$ there exists a a polytope $P \in \mathcal{K}_o^n$ with vertices $r_P(v_i)$ solving the Gauss Image problem and $u_j \in \mathring{\boldsymbol{\alpha}}_K(v_{f(j)})$, then we call f *a solution function*. Sometimes we write $\mathbb{F}_{\mu,\lambda}$ to specify the measures.

The Discrete Gauss Image problem has effectively two steps. The first is a combinatorial problem: what are necessary and sufficient conditions for $\mathbb{F}_{\mu,\lambda} \neq \varnothing$? The second is a geometric problem: does there exist a solution to the Gauss Image problem for a given assignment function?

In this work, we restrict our attention to the equal-weights λ problem ($\forall j\ \lambda_j = 1$). The equal-weights λ problem captures all varieties of a "geometric part" of the Discrete Gauss Image problem, as any problem is reducible to the Discrete Gauss Image problem with equal-weight λ. For a detailed explanation, see *Remark* at the end of the Weak Aleksandrov Condition Section.

The Discrete Gauss Image problem *Suppose λ is a discrete equal-weight ($\forall j\ \lambda_j = 1$) measure on S^{n-1}, and μ is a discrete measure on S^{n-1}. What are the necessary and sufficient* *conditions, on λ and μ, so that there exists a body $K \in \mathcal{K}_o^n$ such that*

$$\mu = \lambda(K, \cdot)? \tag{1.9}$$

And if such body exists, to what extent it is unique?

When μ and λ are simultaneously equal-weight measures, we have a good subclass of problems where the combinatorial step is trivial. In order to guarantee the existence of a solution, the measures should clearly satisfy $\mu(S^{n-1}) = \lambda(S^{n-1})$. Thus, if λ and μ have equal-weights, then they should have the same number of summands. Thus, the assignment functions for these problem are just permutations. Recalling that we can concentrate on polytopes with vertices at the v_i directions, the problem, has a very elementary statement: Given two sets of the same number of unit vectors v_i and u_j, is there a convex polytope P containing the origin with vertices at the v_i directions, such that each normal cone at any vertex $r_P(v_i)$ contains exactly one vector u_j in its interior? This recovers the simple case of the problem established earlier.

In order to attack the problem, we introduce the weaker version of the Aleksandrov condition from [14]. This condition turns out to be a necessary condition for two discrete measures to be related by a convex body. Moreover, if one views the assignment function as a matching in a bipartite graph with vertices $\{u_j\}$ and $\{v_i\}$, this condition is equivalent to assumption in Hall's marriage theorem. See Lemma 4.3.

Definition 1.5. Two discrete measures μ and λ on S^{n-1} are called weak Aleksandrov related if

$$\lambda(S^{n-1}) = \mu(S^{n-1}) \geq \mu(\omega) + \lambda(\omega^*) \tag{1.10}$$

for each compact, spherically convex set $\omega \subset S^{n-1}$.

It turns out that solution to the Discrete Gauss Image problem is equivalent to the following combinatorial problem:

The Assignment Problem *Suppose λ is a discrete equal-weight measure on S^{n-1}, and μ is a discrete measure on S^{n-1}. For each $f \in \mathbb{F}_{\mu,\lambda}$, define*

$$A(f) := \sum_{j=1}^{k} \log u_j v_{f(j)} \tag{1.11}$$

where the log *of negative values is defined as $-\infty$.*

What are the necessary and sufficient conditions, on λ and μ, such that $A(\cdot)$ is maximized by exactly one $f \in \mathbb{F}_{\mu,\lambda}$, and also for this f that $A(f) > -\infty$?

We establish the equivalence of the Discrete Gauss Image problem and the Assignment Problem. That is, given a discrete equal-weight λ and a discrete μ on S^{n-1}, there exists a solution to the Gauss Image Problem if and only if $A(\cdot)$ is uniquely maximized by some $f \in \mathbb{F}_{\mu,\lambda}$ with $A(f) > -\infty$. See Theorem 5.1.

We later proceed to investigate the Assignment Problem. We show that the part about $A(f) > -\infty$ is equivalent to the weak Aleksandrov condition, see Lemma 4.3. We establish the geometric condition which solves the Assignment Problem. The condition is the following:

Edge-normal loop free measures Discrete measures μ and λ are called edge-normal loop free if there does not exist a piecewise linear closed path with at least two vertices such that every vertex is on the different ray $\{tv_i \mid t > 0\}$ and each segment is perpendicular to different u_j. (In the case of two vertices, two edges coincide.)

With this condition we obtain the following:

Theorem 1.6. *Let λ be a discrete equal-weight measure and μ be a discrete measure. Suppose they are weak Aleksandrov related, edge-normal loop free and μ is not concentrated on a closed hemisphere. Then there exist a unique $f \in \mathbb{F}_{\mu,\lambda}$ which maximizes $A(\cdot)$. In particular, there exist a polytope $P \in \mathcal{K}_o^n$ with vertices $r_P(v_i)$ solving the Discrete Gauss Image Problem:*

$$\mu = \lambda(K, \cdot) \tag{1.12}$$

We also investigate the Assignment Problem and the Discrete Gauss Image Problem from generic point of view where we obtain:

Theorem 1.7. *Suppose we are given $m \in \mathbb{N}$ and coefficients $\mu_i \in \mathbb{N}$. Consider all pairs of discrete measures μ and discrete equal-weight measures λ satisfying the weak Aleksandrov condition, such that μ is not concentrated on a closed hemisphere. Then for a generic measure in this class, there exists a polytope $P \in \mathcal{K}_o^n$ with vertices $r_P(v_i)$ satisfying*

$$\mu = \lambda(P, \cdot). \tag{1.13}$$

Since the solution to the Discrete Gauss Image problem is always non-unique if it exists, see Proposition 5.12, the proper reformulation for the uniqueness question is the uniqueness of the assignment functional. To this extent, we show that the assignment functional is unique. That is, if $K, L \in \mathcal{K}_o^n$ satisfy $\mu = \lambda(K, \cdot) = \lambda(K, \cdot)$, then $u_j \in \overset{\circ}{\boldsymbol{\alpha}}_K(v_i)$ if and only if $u_j \in \overset{\circ}{\boldsymbol{\alpha}}_L(v_i)$.

Let us make some remarks regarding the proofs. In many analogues and variations of Minkowski problems, one can usually construct a functional on the set of convex bodies such that body which maximizes the functional turns out to be the solution. For these problems, it is usually easy to verify that the convex body maximizing the functional is a solution to the given problem, yet it is usually sufficiently harder to show the existence of the body maximizing the functional. Such functional, see Equation (2.13), is indeed applicable to this problem. Yet, something opposite happens: it is easy to see that there exists a body maximizing the functional, but the maximizing body is not necessarily a solution, contrary to the case of when λ is absolutely continuous in [14], for example. The difficulty lies in the interior part of the condition $u_j \in \overset{\circ}{\boldsymbol{\alpha}}_K(v_j)$.

We give two different proofs for the equivalence between the Discrete Gauss Image problem and the Assignment Problem. The main proof has the following structure. First we construct a sequence of absolutely continuous measures $\lambda_\varepsilon \to \lambda$, which are weak Aleksandrov related to μ. Then we show the existence of a solution K_ε for the λ_ε, μ-problem under the weak Aleksandrov assumption. We then study the subsequence convergence of K_ε and find conditions on when the limiting body is the solution to the original question. The last part relies on a generalization of the Birkhoff-von Neumann theorem and transportation polytopes. The second proof, for a special case of when both measures are equal-weight is given in section 6. It is shorter but more computational. It uses new computational results from recent works of Wyczesany on mass transport [47]. The connection here is not unexpected, as one can reformulate the Gauss Image Problem as a mass transport problem. Moreover, the results in [47] are advances (besides other things) on the well-known cyclic monotonicity condition introduced by Rockafellar in [40].

Finally, the main proof relies on the existence of the solution to the Gauss Image problem for when λ is an absolutely continuous measure that is weak Aleksandrov related to the discrete measure μ. This, more technical result, has been moved to our second paper to not distract the reader from the main focus of this work. We keep the statements of this and similar results from our second paper in the Appendix for convenience. Yet, they present an interest by themselves, as they use a necessary assumption for measures to be related by a convex body. The importance of the weak Aleksandrov relation compared to its strong counterpart comes from the necessity of this assumption for the discrete problem and many examples of discrete measures related by a convex body which satisfies only the weak version of Aleksandrov relation. The discussion of different versions of Aleksandrov relations are in Section 3.

2. Preliminaries

Let $\mathcal{K}^n$ be the set of convex bodies in $\mathbb{R}^n$, that is compact convex sets with nonempty interior. By $\mathcal{K}^n_o \subset \mathcal{K}^n$, we denote those convex bodies which contain origin in their interior. By ∂K, we denote the boundary of K. Given $K \in \mathcal{K}^n_o$, *the radial map* $r_K : S^{n-1} \to \partial K$ is defined by

$$r_K(u) = ru \in \partial K, \tag{2.1}$$

for some positive r. By $N(K, x)$, we denote the *normal cone of* K *at* $x \in \partial K$, that is the set of all outer unit normals at x:

$$N(K, x) = \{v \in S^{n-1} : (y - x) \cdot v \leq 0 \text{ for all } y \in K\}. \tag{2.2}$$

Given $K \in \mathcal{K}^n_o$, we define the radial Gauss image of $\omega \subset S^{n-1}$ as:

$$\boldsymbol{\alpha}_K(\omega) = \bigcup_{x \in r_K(\omega)} N(K, x) \subset S^{n-1}. \tag{2.3}$$

The radial Gauss image $\boldsymbol{\alpha}_K$ maps sets of S^{n-1} into sets of S^{n-1}. Outside of a spherical Lebesgue measure zero set, multivalued map $\boldsymbol{\alpha}_K$ is singular valued. It is known that $\boldsymbol{\alpha}_K$ maps Borel measurable sets into Lebesgue measurable sets. See [43] for both of these results. We denote the restriction of $\boldsymbol{\alpha}_K$ to be the singular value map by α_K. For details, see [14].

Suppose λ is a submeasure on S^{n-1} defined on spherical Lebesgue measurable sets and $K \in \mathcal{K}^n_o$. Then $\lambda(K, \cdot)$, *the Gauss image measure of* λ *via* K, is a submeasure defined as the pushforward of λ via the map $\boldsymbol{\alpha}_K$. That is, for each Borel $\omega \subset S^{n-1}$,

$$\lambda(\boldsymbol{\alpha}_K(\omega)) = \lambda(K, \omega). \tag{2.4}$$

If λ is absolutely continuous, $\lambda(K, \cdot)$ is a Borel measure [14].

The radial function $\rho_K : S^{n-1} \to \mathbb{R}$ is defined by:

$$\rho_K(u) = \max\{a : au \in K\}. \tag{2.5}$$

In this case, $r_K(u) = \rho_K(u)u$. *The support function* is defined by

$$h_K(x) = \max\{x \cdot y : y \in K\}. \tag{2.6}$$

For $K \in \mathcal{K}^n_o$ we define its *polar body* K^* by $h_{K^*} := \frac{1}{\rho_K}$. We denote by r_k the radius of the largest ball contained in K centered at o and by R_k the radius of the smallest ball containing

K centered at o as well. Clearly, $r_k \leq R_k$. *The support hyperplane* to K with outer unit normal $v \in S^{n-1}$ is defined by

$$H_K(v) = \{x : x \cdot v = h_K(v)\}. \tag{2.7}$$

We define $H^-(\alpha, v) = \{x : x \cdot v \leq \alpha\}$.

For $\omega \subset S^{n-1}$, we define $\operatorname{cone}\omega \subset \mathbb{R}^n$, the *cone that* ω *generates*, as

$$\operatorname{cone}\omega = \{tu : t \geq 0 \text{ and } u \in \omega\}. \tag{2.8}$$

And define $\hat{\omega}$, the *restricted cone that* ω *generates*, as

$$\hat{\omega} = \{tu : 0 \leq t \leq 1 \text{ and } u \in \omega\}. \tag{2.9}$$

We are going to say that $\omega \subset S^{n-1}$ is *spherically convex* if the cone that ω generates is a nonempty convex subset of $\mathbb{R}^n$, that is not all of $\mathbb{R}^n$. Therefore, a spherically convex set on S^{n-1} is nonempty and is always contained in a closed hemisphere of S^{n-1}. For a subset $\omega \subset S^{n-1}$ which is contained in a closed hemisphere, we define the *spherical convex hull*, $\langle\omega\rangle$, of ω, by

$$\langle\omega\rangle = S^{n-1} \cap \operatorname{conv}(\operatorname{cone}\omega). \tag{2.10}$$

where conv stands for the convex hull in $\mathbb{R}^n$. Given $\omega \subset S^{n-1}$ contained in a closed hemisphere, *polar set* ω^* is defined by:

$$\omega^* = \bigcap_{u \in \omega} \{v \in S^{n-1} : u \cdot v \leq 0\}. \tag{2.11}$$

We note that polar set is always convex. If $\omega \subset S^{n-1}$ is a closed set, we define its *outer parallel set* ω_α, as

$$\omega_\alpha = \bigcup_{u \in \omega} \{v \in S^{n-1} : u \cdot v > \cos\alpha\}. \tag{2.12}$$

For recent work on spherical convex bodies, see Besau and Werner [10].

Given $K \in \mathcal{K}_o^n$ and measures μ and λ, we define the functional $\Phi(K, \mu, \lambda)$ by

$$\Phi(K, \mu, \lambda) := \int \log\rho_K d\mu + \int \log\rho_{K^*} d\lambda. \tag{2.13}$$

Note that $\Phi(K, \mu, \lambda) = \Phi_{\mu,\lambda}(K^*)$ in [14] notation.

We call measures λ and μ on S^{n-1} *discrete* if they have the form:

$$\begin{aligned} \lambda &= \sum_{j=1}^{k} \lambda_j \delta_{u_j} \\ \mu &= \sum_{i=1}^{m} \mu_i \delta_{v_i}, \end{aligned} \tag{2.14}$$

where δ is the Dirac measure and $\lambda_j, \mu_i > 0$. Discrete measure λ is called *equal-weight* if $\lambda_j = 1$ for all j. Similarly, for μ. Note that we only deal with finitely many weights, as this is our main interest. The notation is consistent: j, u_j, λ_j, k are always associated with λ and i, v_i, μ_i, m are always associated with μ. Given a discrete measure μ not concentrated on closed hemisphere, we define $\mathcal{P}_\mu \subset \mathcal{K}_o^n$ as the set of polytopes containing the origin of the form

$$P = (\bigcap_{i=1}^{m} H^-(\alpha_i, v_i))^*, \tag{2.15}$$

where $\alpha_i > 0$. That is, $\mathcal{P}_\mu$ is the set of all convex polytopes containing the origin with vertices $r_P(v_i)$.

The Gauss image measure was defined in Introduction. We just note that if for a given μ and λ, there exists $K \in \mathcal{K}_o^n$ such that $\mu = \lambda(K, \cdot)$, we say that measures μ and λ are related by convex body K. By $\bar{\lambda}$, we denote the Lebesgue measure on S^{n-1}.

We use the books of Schneider [43] as our standard reference. The books of Gruber and Gardner are also good alternatives [18, 19].

3. Weak Aleksandrov Condition

Let μ and λ be discrete equal-weight measures. Suppose there exists a solution $P \in \mathcal{K}_o^n$ such that $\mu = \lambda(P, \cdot)$. Clearly, this implies $m = \mu(S^{n-1}) = \lambda(S^{n-1}) = k$. Note that if $u \in \{u_1, \ldots, u_m\}$ is contained in normal cone of a vertex at direction $v \in \{v_1, \ldots, v_m\}$, then $uv > 0$ since $P \in \mathcal{K}_o^n$. So, before we attempt to find the solution for the given measures, we should guarantee the existence of a pairing between two sets of vectors such that $uv > 0$ in each pair (see Proposition 3.4). This leads us two to questions: (1) Does there exist a good necessary assumption that guarantees the existence of a pairings between vectors $\{u_1 \ldots u_m\}$ and $\{v_1 \ldots v_m\}$ such that $uv > 0$ for each pair? and (2) For a specified pairing, does there exist a solution? The answer to the first part turns out to be the weak Aleksandrov condition.

The Aleksandrov condition (Defenition 1.2) and the weak Aleksandrov condition (Definition 1.5) were stated in the Introduction. Note that we use the phrases *"Aleksandrov condition"* and *"Aleksandrov relation"* interchangeably. Since for any $\omega \subset S^{n-1}$ a compact spherically convex set, $S^{n-1} \setminus \omega^* = \omega_{\frac{\pi}{2}}$ and $\omega^{**} = \omega$, we immediately obtain the following equivalent definitions:

Proposition 3.1. *Two Borel measures μ and λ on S^{n-1} are Aleksandrov related if and only if $\mu(S^{n-1}) = \lambda(S^{n-1})$, and for each compact spherically convex set $\omega \subset S^{n-1}$*

$$, \mu(\omega) < \lambda(\omega_{\frac{\pi}{2}}). \tag{3.1}$$

Remark. Sometimes we will write Strong Aleksandrov related in place of Aleksandrov related to emphasize the difference. △

Proposition 3.2. *Two discrete Borel measures μ and λ on S^{n-1} are weak Aleksandrov related if and only if $\mu(S^{n-1}) = \lambda(S^{n-1})$, and for each compact spherically convex set $\omega \subset S^{n-1}$*

$$, \mu(\omega) \leq \lambda(\omega_{\frac{\pi}{2}}). \tag{3.2}$$

Or, alternatively, for each compact spherically convex set $\omega \subset S^{n-1}$,

$$\lambda(\omega) \leq \mu(\omega_{\frac{\pi}{2}}). \tag{3.3}$$

The difference between the two conditions is demonstrated in the following example:

Example 3.3. Take any equilateral triangle P in $\mathbb{R}^2$ centered at the origin with vertices on the unit sphere. Let v_1, v_2, v_3 be such that $r_P(v_i) = v_i$ are different vertices of the triangle. Let $\mu = \lambda$ be the discrete equal-weights measure:

$$\lambda = \mu = \sum_{i=1}^{3} \delta_{v_i}. \tag{3.4}$$

Then, clearly, $\mu = \lambda(P, \cdot)$. Note that μ and λ are weak Aleksandrov related but not strong Aleksandrov related. It is also interesting to note that for these particular measures, any triangle $P \in \mathcal{P}_\mu$ is a solution. We leave the details to the reader.

The next proposition provides a necessary condition for two measures to be related by a convex body. In particular, it shows that the weak Aleksandrov relation is a necessary condition for two discrete measures to be related by a convex body.

Proposition 3.4. *Given two Borel measures μ and λ on S^{n-1}, suppose they are related by a convex body $K \in \mathcal{K}_o^n$. That is, $\mu = \lambda(K, \cdot)$. Then $\mu(S^{n-1}) = \lambda(S^{n-1})$, and there exists a uniform $\alpha > 0$ such that for each compact, spherically convex set $\omega \subset S^{n-1}$,*

$$\mu(\omega) \leq \lambda(\omega_{\frac{\pi}{2}-\alpha}). \tag{3.5}$$

Moreover, if μ and λ are discrete, then they are weak Aleksandrov related.

Proof. Since $K \in \mathcal{K}_o^n$, there exists a $c > 0$ such that $\frac{r_K}{R_K} > c$. Consider some $u \in S^{n-1}$ and $v \in \boldsymbol{\alpha}_K(u)$. Then

$$r_K \leq h_K(v) = \rho_K(u)uv \leq R_K uv. \tag{3.6}$$

Hence, $c < \frac{r_K}{R_K} \leq uv$. So, for each $u \in S^{n-1}$, we have

$$\boldsymbol{\alpha}_K(u) \subset u_{\arccos(c)} \subset u_{\frac{\pi}{2}-\alpha}, \tag{3.7}$$

for some α where $0 < \alpha < \frac{\pi}{2}$. Therefore, for any compact spherically convex ω, since $\omega_{\frac{\pi}{2}-\alpha} = \cup_{u\in\omega} u_{\frac{\pi}{2}-\alpha}$, we obtain

$$\mu(\omega) = \lambda(K, \omega) = \lambda(\boldsymbol{\alpha}_K(\omega)) \leq \lambda(\omega_{\frac{\pi}{2}-\alpha}). \tag{3.8}$$

The second part of the claim immediately follows from Proposition 3.2. □

We also state this lemma as a trivial consequence of the above proof for later reference.

Lemma 3.5. *Given $K \in \mathcal{K}_o^n$, suppose $0 < c < \frac{r_K}{R_k}$. Then for any $v \in S^{n-1}$,*

$$\boldsymbol{\alpha}_K(v) \subset v_{\arccos c}. \tag{3.9}$$

Note that for discrete measures, the weak Aleksandrov relation actually implies the above conclusion since every compact spherically convex set is closed:

Proposition 3.6. *Suppose the discrete measures μ and λ satisfy the weak Aleksandrov relation. Then there exists a uniform $\alpha > 0$ such that for each closed set $\omega \subset S^{n-1}$:*

$$\mu(\omega) \leq \lambda(\omega_{\frac{\pi}{2}-\alpha}). \tag{3.10}$$

Proof. First we claim that for each closed set $\omega \subset S^{n-1}$ contained in a closed hemisphere, $\omega_{\frac{\pi}{2}} = \langle\omega\rangle_{\frac{\pi}{2}}$. By set inclusion, $\omega_{\frac{\pi}{2}} \subset \langle\omega\rangle_{\frac{\pi}{2}}$. For the opposite direction, take any $v \in \langle\omega\rangle_{\frac{\pi}{2}}$. Then for some $x \in \langle\omega\rangle$, we have that $xv > 0$. By the definition of convex hull, we can write x as some convex combination of finite number of vectors $y_j \in \text{cone}(\omega)$. By the definition of the cone, each y_j is a positive scaling of some vector $z_j \in \omega$. Hence, x is a linear combination of some vectors z_j with positive coefficients. Therefore, since $xv > 0$, we have that for some z_j, $z_j v > 0$, and thus $v \in \omega_{\frac{\pi}{2}}$. So, $\omega_{\frac{\pi}{2}} \supset \langle\omega\rangle_{\frac{\pi}{2}}$. And hence, $\omega_{\frac{\pi}{2}} = \langle\omega\rangle_{\frac{\pi}{2}}$.

Let $\mathcal{A}$ be a collection of all possible indices $I \in \{1 \dots m\}$ such that $\{v_i\}_{i\in I}$ are contained in closed hemisphere. Given $I \in \mathcal{A}$, define ω^I as $\cup_{i\in I} v_i$. Then by Proposition 3.2 and from the previous conclusion, we obtain:

$$\mu(\omega^I) \leq \mu(\langle\omega^I\rangle) \leq \lambda(\langle\omega^I\rangle_{\frac{\pi}{2}}) = \lambda({\omega^I}_{\frac{\pi}{2}}). \tag{3.11}$$

Since λ is a discrete measure and $(v_i)_{\frac{\pi}{2}}$ is an open set, for any v_i, where $i \in \{1 \dots m\}$, there exists an α_i such that $\lambda((v_i)_{\frac{\pi}{2}-\alpha_i}) = \lambda((v_i)_{\frac{\pi}{2}})$. Let $\alpha = \min_i \alpha_i$. Therefore, by the definition of outer parallel set:

$$\lambda({\omega^I}_{\frac{\pi}{2}} \setminus {\omega^I}_{\frac{\pi}{2}-\alpha}) = \lambda(\bigcup_{i\in I}(v_i)_{\frac{\pi}{2}} \setminus \bigcup_{i\in I}(v_i)_{\frac{\pi}{2}-\alpha}) \leq \lambda(\bigcup_{i\in I}((v_i)_{\frac{\pi}{2}} \setminus (v_i)_{\frac{\pi}{2}-\alpha_i})) = 0, \tag{3.12}$$

where the middle inequality follows from the set inclusion. Combining the last two equations, we obtain that for any $I \in \mathcal{A}$,

$$\mu(\omega^I) \leq \lambda(\omega^I_{\frac{\pi}{2}-\alpha}). \tag{3.13}$$

Suppose now we are given an index set $I \in \{1 \dots m\}$ such that $\{v_i\}_{i\in I}$ are not contained in a closed hemisphere. Then $\mu(\omega^I) \leq \mu(S^{n-1}) = \lambda(S^{n-1}) = \lambda(\omega^I_{\frac{\pi}{2}})$. And similarly to (3.12), we obtain that $\lambda(\omega^I_{\frac{\pi}{2}}) = \lambda(\omega^I_{\frac{\pi}{2}-\alpha})$. Therefore, combining with the previous claim, we obtain that:

$$\mu(\omega^I) \leq \lambda(\omega^I_{\frac{\pi}{2}-\alpha}) \tag{3.14}$$

for any index set $I \in \{1 \dots m\}$.

Now given any closed set $\omega \subset S^{n-1}$, let $\omega^I = \omega \cap \{v_1 \dots v_m\}$. Then from the previous inequality,

$$\mu(\omega) = \mu(\omega^I) \leq \lambda(\omega^I_{\frac{\pi}{2}-\alpha}) \leq \lambda(\omega_{\frac{\pi}{2}-\alpha}), \tag{3.15}$$

where the last inequality follows by the set inclusion. □

4. The Assignment Functional

In this section, we introduce the assignment functional, which is related to the existence of a solution to the Discrete Gauss Image problem. We will first show that it suffices to reduce the problem to polytopes.

Proposition 4.1. *Given discrete measure μ not concentrated on a closed hemisphere and discrete measure λ, suppose there exists a body $K \in \mathcal{K}^n_o$ satisfying $\mu = \lambda(K, \cdot)$. Let $P \in \mathcal{P}_\mu$ be a polytope with vertices $r_K(v_i)$,*

$$P = (\bigcap_{i=1}^{m} H^-(1/r_K(v_i), v_i))^*. \tag{4.1}$$

Then $\mu = \lambda(P, \cdot)$, and for each j, $u_j \in \boldsymbol{\alpha}_K(\overset{\circ}{v}_{f(j)})$, for some function $f\colon \{1 \dots k\} \to \{1 \dots m\}$.

Proof. The existence of a polytope P is justified by simply taking the convex hull of points $r_k(v_i)$. Since μ is not concentrated on a closed hemisphere, we obtain that P contains the origin in its interior. Hence, $P \in \mathcal{K}_o^n$ and $P \in \mathcal{P}_\mu$.

First we claim that for any $i_1 \neq i_2$, the set $\boldsymbol{\alpha}_K(v_{i_1}), \cap \boldsymbol{\alpha}_K(v_{i_2})$ does not contain any vectors $\{u_1 \dots u_k\}$. We proceed by contradiction and suppose the claim is not true. Let some $u_j \in \boldsymbol{\alpha}_K(v_{i_1}) \cap \boldsymbol{\alpha}_K(v_{i_2})$. First note that $v_{i_1} \neq -v_{i_2}$, as for any $v \in S^{n-1}$:

$$\boldsymbol{\alpha}_K(v) \subset v_{\frac{\pi}{2}}, \tag{4.2}$$

from Lemma 3.5. And thus $\boldsymbol{\alpha}_K(v) \cap \boldsymbol{\alpha}_K(-v) = 0$. Note that the set of boundary points of K for which u_j is normal,

$$H_K(u_j) \cap K \tag{4.3}$$

is a convex set. Hence,

$$\text{cone}\,(H_K(u_j) \cap K) = \{tx : t \geq 0 \text{ and } x \in H_K(u_j) \cap K\} \tag{4.4}$$

is a convex set. So $r_K^{-1}(H_K(u_j) \cap K)$ is a spherical convex set, as $\text{cone}\,(H_K(u_j) \cap K) \neq \mathbb{R}^n$. Since $v_{i_1}, v_{i_2} \in r_K^{-1}(H_K(u_j) \cap K)$, we obtain $\langle\{v_{i_1}, v_{i_2}\}\rangle \subset r_K^{-1}(H_K(u_j) \cap K)$. Therefore, for any $v \in \langle\{v_{i_1}, v_{i_2}\}\rangle$, we have $u_j \in \boldsymbol{\alpha}_K(v)$. Thus $\lambda(K, v) \geq 1$ for uncountably many $v \in \langle\{v_{i_1}, v_{i_2}\}\rangle$, and therefore, $\lambda(K, \cdot)$ can't be a finite measure. We arrive at a contradiction.

Therefore, given j, we can properly define a function f by $u_j \in \boldsymbol{\alpha}_K(v_{f(j)})$. Since for any i, $\boldsymbol{\alpha}_K(v_i) \subset \boldsymbol{\alpha}_P(v_i)$, we obtain that $u_j \in \boldsymbol{\alpha}_P(v_{f(j)})$. We now check that the interior assumption is satisfied. That is, $u_j \in \boldsymbol{\alpha}_P(\overset{\circ}{v}_{f(j)})$, which will finish the proof. Suppose it is not. Pick some $u_j \in \partial \boldsymbol{\alpha}_P(v_{f(j)})$. Since P is a polytope with vertex $r_P(v_{f(j)})$, there exist some other v_i for $i \neq f(j)$, such that $u_j \in \partial \boldsymbol{\alpha}_P(v_i)$. Now, for the support function h_P, we obtain

$$h_P(u_j) = u_j r_P(v_i) = u_j r_P(v_{f(j)}). \tag{4.5}$$

Since we showed that $u_j \notin \boldsymbol{\alpha}_K(v_{i_1}) \cap \boldsymbol{\alpha}_K(v_{i_2})$ for any $i_1 \neq i_2$, we have that $u_j \notin \boldsymbol{\alpha}_K(v_i)$. Therefore,

$$u_j r_P(v_i) = u_j r_K(v_i) < h_K(u_i), \tag{4.6}$$

where we additionally used that $r_K(v_i) = r_P(v_i)$. However, since $u_j \in \boldsymbol{\alpha}_K(v_{f(j)})$, we have

$$u_j r_P(v_{f(i)}) = u_j r_K(v_{f(i)}) = h_K(u_i). \tag{4.7}$$

We get the contradiction from the combination of the last three equations. □

Parts of next definition were defined in the Introduction section.

Definition 4.2. Given discrete measure λ and discrete measure μ, we associate *the set of assignment functions*:

$$\mathbb{F}_{\mu,\lambda} := \Big\{ f \colon \{1 \dots k\} \to \{1 \dots m\} \mid \sum_{j \in f^{-1}(i)} \lambda_j = \mu_i \Big\}. \tag{4.8}$$

We call $f \in \mathbb{F}_{\mu,\lambda}$ *an assignment function* with respect to μ and λ. If for $f \in \mathbb{F}_{\mu,\lambda}$ there exists a body $K \in \mathcal{K}_o^n$ solving the Gauss Image problem for measures μ and λ and $u_j \in \boldsymbol{\alpha}_K(\overset{\circ}{v}_{f(j)})$, then we call f *a solution function.* We also say that f, or f_K, is an assignment function with respect to the body K. Sometimes we will write $\mathbb{F}$ instead of $\mathbb{F}_{\mu,\lambda}$. Finally, we define the set of the proper assignment functions as a subset of the above:

$$\mathbb{F}_{\mu,\lambda,p} := \{f \in \mathbb{F} \mid \forall j, u_j v_{f(j)} > 0\}. \tag{4.9}$$

Sometimes we write $\mathbb{F}_p$ instead of $\mathbb{F}_{\mu,\lambda,p}$.

Remark. Note that if $f \in \mathbb{F}_p \setminus \mathbb{F}$, then f can not be a solution function. △

We claim that if measures satisfy the weak Aleksandrov relation, then $\mathbb{F}_p \neq \varnothing$. There are several ways to see this. First, would be the consequence of the proof of the main Theorem 5.1. Alternatively, one can prove the statement by induction. Here, we present a proof for equal-weight μ and λ, which provides an interesting connection to the graph theory and gives another justification for the weak Aleksandrov relation.

Suppose we are given two discrete equal-weight measures μ and λ. Consider a bipartite graph $G(\mu, \lambda)$ formed by two sets of vertices $\{u_j\}$ and $\{v_i\}$ where we draw an edge from v_i to u_j if and only if $v_i u_j > 0$. For $V \subset \{v_1 \dots v_m\}$ define $N_G(V)$ as the set of adjacent verticies to V. Hall's marriage theorem states that there exist a perfect matching between verticies of $G(\mu, \lambda)$ if and only if for each subset of vertices $V \subset \{v_1 \dots v_m\}$:

$$|V| \leq |N_G(V)|. \tag{4.10}$$

Notice that this is exactly the same as weak Aleksandrov condition:

Lemma 4.3. *Suppose we are given two discrete equal-weight measures μ and λ, where $\lambda(\mathbb{S}^{n-1}) = \mu(\mathbb{S}^{n-1})$. Then Hall's marriage condition is equivalent to the weak Aleksandrov condition. In particular, if two measures satisfy weak Aleksandrov condition, then $\mathbb{F}_p$ is nonempty.*

Proof. Suppose for each subset of vertices $V \subset \{v_1 \dots v_m\}$,

$$|V| \leq |N_G(V)|. \tag{4.11}$$

Then for any ω a compact spherically convex set,

$$\begin{aligned} \mu(\omega) = &|\omega \cap \{v_1 \dots v_m\}| \leq |N_G(\omega \cap \{v_1 \dots v_m\})| \leq \\ &|\omega_{\frac{\pi}{2}} \cap \{u_1 \dots u_m\}| = \lambda(\omega_{\frac{\pi}{2}}). \end{aligned} \tag{4.12}$$

Thus, by Proposition 3.2, μ and λ are weak Aleksandrov related. For another direction, suppose the weak Aleksandrov condition holds. Consider any $V \subset \{v_1 \dots v_m\}$. Then V is a closed set, and by Proposition 3.6, $\mu(V) \leq \lambda(V_{\frac{\pi}{2}-\alpha})$ for some constant α. Since $\lambda(V_{\frac{\pi}{2}-\alpha}) \leq |N_G(V)|$, we obtain that $|V| \leq |N_G(V)|$.

We just proved equivalence of the two conditions. Given the weak Aleksandrov assumption, Hall's marriage Theorem tells us that the graph formed from its vertices has a matching. That is, there exists a σ permutation such that $v_j u_{\sigma(j)} > 0$ for all j. Thus, $\mathbb{F}_p \neq \varnothing$. □

Remark. In some sense, all possible Aleksandrov relations, even for the non-discrete problem, are related to Hall's Marriage Theorem. It is interesting to note that the theorems related to the Discrete Gauss Image problem provided in the next sections actually prove Hall's marriage theorem for specific bipartite graphs $G(\mu, \lambda)$. If we show existence of a convex body $K \in \mathcal{K}_o^n$ such that $\mu = \lambda(K, \cdot)$, then the assignment function with respect to this body gives a perfect matching. In some sense, the results of the next section present a functional approach to Hall's marriage theorem. △

Definition 4.4. For each $f \in \mathbb{F}$, we define *the assignment functional*

$$A(f) := \sum_{j=1}^{k} \log u_j v_{f(j)}, \tag{4.13}$$

where the log of negative values is defined to be $-\infty$.

Note that $f \in \mathbb{F}_p$ if and only if $A(f) > -\infty$. The $-\infty$ in the definition does not create a problem since $u_j v_{f(j)} \leq 1$ for any vectors. That is, we never sum $-\infty$ with $+\infty$. In the next section, we will show that as long as the assignment functional is uniquely maximized on the set $\mathbb{F}_{\mu,\lambda}$, and $\mathbb{F}_p \neq \varnothing$, there exists a polytope $P \in \mathcal{P}_\mu$ satisfying $\mu = \lambda(P, \cdot)$. On the other hand, if the maximizer is non-unique, then we show that the solution does not exist.

Remark. We would like to comment on the remark made in the beginning about restricting to equal-weight measures λ. The motivation comes from the following. Suppose for given discrete measures μ and λ, we have $\mathbb{F}_{\mu,\lambda} \neq \varnothing$. Let $f \in \mathbb{F}_{\mu,\lambda}$. Then we can define new discrete measures λ' and μ' to be

$$\begin{aligned} \lambda' &:= \sum_{j=1}^{k} \delta_{u_j} \\ \mu' &= \sum_{i=1}^{m} \mu'_i \delta_{v_i} := \sum_{i=1}^{m} |f^{-1}(i)| \delta_{v_i}, \end{aligned} \tag{4.14}$$

where $|f^{-1}(i)|$ denotes the number of elements mapping to i. It is easy to see that $f \in \mathbb{F}_{\mu',\lambda'}$. If there exists a $K \in \mathcal{K}_o^n$ solving $\mu' = \lambda'(K, \cdot)$ such that $u_j \in \boldsymbol{\alpha}_K(\mathring{v}_{f(j)})$, then K automatically solves the original problem $\mu = \lambda(K, \cdot)$ and vice versa. Therefore, f is a solution function for discrete μ, λ problem if and only if f is a solution function for discrete μ', λ' problem where λ' is equal-weight. Hence the equal-weights λ problems capture all varieties of a "geometric part" of the Discrete Gauss Image problem, as any problem is reducible to the Discrete Gauss Image problem with equal-weights λ.

$\triangle$

5. Existence and Uniqueness

In this section, we will prove Theorem 5.1, which presents an equivalence between the Discrete Gauss Image problem and The Assignment Problem. We note that for convenience, we will assume that μ is not concentrated on a closed hemisphere. It seems that nothing prevents similar results to hold without this restriction. Roughly, this happens because if μ is not concentrated on a closed hemisphere, then by considering some body K, one can start to decrease this body in radial directions $\rho_K(u)$ where $uv_i \leq 0$ increasing each $\boldsymbol{\alpha}_K(v_i)$. However, we do not proceed in this direction, since by assuming that μ is not concentrated on a closed hemisphere, we can speak of a class of polytopes P_μ that has a natural resemblance with the discrete version of classical Minkowski problem.

Theorem 5.1. *Let λ be a discrete equal-weight measure and μ be a discrete measure. Suppose they are weak Aleksandrov related and μ is not concentrated on a closed hemisphere. Then $\mathbb{F}_p$ is nonempty. Moreover, $f \in \mathbb{F}$ is a solution function if and only if it is the unique maximizer of the assignment functional. In other words,*

- *The assignment functional, $A(f)$, is maximized at exactly one $\overset{\circ}{f} \in \mathbb{F}$. For this f, there exists a polytope $P \in \mathcal{P}_\mu$ such that $\lambda(P, \cdot) = \mu$ and $u_j \in \boldsymbol{\alpha}_P(v_{f(j)})$.*
- *Or $A(f)$ is maximized at more than one $f \in \mathbb{F}$, in which case there is no convex body $K \in \mathcal{K}_o^n$ such that $\lambda(K, \cdot) = \mu$.*

Recall the functional $\Phi(K, \mu, \lambda)$ defined in (2.13). We start with some preliminary propositions.

Proposition 5.2. *Let λ and μ be discrete measures. Suppose they are weak Aleksandrov related. Suppose also that μ is not concentrated on a closed hemisphere. Then, for any assignment function $f \in \mathbb{F}$ and any $K \in \mathcal{K}_o^n$, we have:*

$$\Phi(K, \mu, \lambda) \leq -A(f). \tag{5.1}$$

Proof. Given any $K \in \mathcal{K}_o^n$, let $P \in \mathcal{P}_\mu$ be a convex polytope with vertices $r_P(v_i) = r_K(v_i)$. Then $P \subset K$, and hence $h_P \leq h_k$, which implies that $\int \log \rho_{K^*} d\lambda \leq \int \log \rho_{P^*} d\lambda$. At the same time, $\int \log \rho_K d\mu = \int \log \rho_P d\mu$ since we preserved the radial distance at the point masses of μ. Hence, $\Phi(K, \mu, \lambda) \leq \Phi(P, \mu, \lambda)$. Recall that we can write P as $P^* = \bigcap_{i=1}^m H^-(1/\alpha_i, v_i)$, where $\alpha_i = \rho_K(v_i)$.

Note that $\frac{1}{\alpha_i v_i u_j}$ is the distance from the center to the intersection between a ray in the u_j direction starting at the center and the hyperplane $H(1/\alpha_i, v_i)$. From this, we obtain that

$$\log \rho_{P^*}(u_j) = \min_{i=1}^{m} \log(\frac{1}{\alpha_i v_i u_j}), \tag{5.2}$$

where we assume that $\log(\frac{1}{x}) = \infty$ if $x < 0$. This relates to the fact that a ray in the u_j direction from the center never intersects the hyperplane $H(1/\alpha_i, v_i)$. Note that by Proposition 3.2,

$$1 = \mu(\langle u_j \rangle) \leq \lambda(\langle u_j \rangle_{\frac{\pi}{2}}). \tag{5.3}$$

So, there always exists some v_i such that $u_j v_i > 0$, and the right hand-side of equation (5.2) is not ∞. Using (2.13), we write:

$$\Phi(P, \mu, \lambda) = \sum_{i=1}^{m} \mu_i \log \alpha_i + \sum_{j=1}^{k} \lambda_j \min_{i=1}^{m} \log(\frac{1}{\alpha_i v_i u_j}). \tag{5.4}$$

So in particular, again assuming $\log(\frac{1}{x}) = \infty$ if $x < 0$, given any assignment function f we have:

$$\Phi(P, \mu, \lambda) \leq \sum_{i=1}^{m} \mu_i \log \alpha_i + \sum_{j=1}^{k} \lambda_j \log(\frac{1}{\alpha_{f(j)} v_{f(j)} u_j}). \tag{5.5}$$

By definition of the assignment function,

$$\sum_{j \in f^{-1}(i)} \lambda_j = \mu_i. \tag{5.6}$$

Therefore, looking back at (5.5), we obtain that α_i on the right side cancels out, which makes it equal to $-A(f)$. Combining the last inequality with $\Phi(K, \mu, \lambda) \leq \Phi(P, \mu, \lambda)$, we prove the claim. $\square$

We easily obtain the following uniqueness result as a corollary of the above argument:

Proposition 5.3 (Uniqueness). *If $g \in \mathbb{F}_{\mu,\lambda}$ is a solution function where λ is an equal-weight measure, and μ is not concentrated on a closed hemisphere, then g is the maximizer of the assignment functional.*

Proof. Since $g \in \mathbb{F}$ is a solution function, let $P \in \mathcal{P}_\mu$ be a polytope solution to the Gauss Image problem such that $\mu = \lambda(P, \cdot)$ and $u_j \in \boldsymbol{\alpha}_K(\mathring{v}_{g(j)})$, which is guaranteed by Proposition 4.1. Since we additionally have:

$$\sum_{j \in f^{-1}(i)} \lambda_j = \mu_i \tag{5.7}$$

and since $\lambda_j = 1$ by the equal-weight assumption, Equation (5.4) in the above Proposition 5.2 gives us that $\Phi(P, \mu, \lambda) = -A(g)$. Combining this with the result of the same proposition, we obtain that $-A(g) \leq -A(f)$ for any $f \in \mathbb{F}$.

$\square$

Define $d\lambda_\varepsilon := \sum_{j=1}^{k} \phi_{\varepsilon,j} d\bar{\lambda}$, where $\bar{\lambda}$ is the Lebesgue measure on the sphere and $\phi_{\varepsilon,i}$ is a bump function taking only two values (0 and ε^{-1}) with disk support centered at u_j of volume ε.

Lemma 5.4. *Let μ and λ be discrete and weak Aleksandrov related measures. Then there exists $\varepsilon' > 0$ and $\alpha > 0$ such that for all $\varepsilon < \varepsilon'$, we have that for each closed set $\omega \subset S^{n-1}$:*

$$\mu(\omega) \leq \lambda_\varepsilon(\omega_{\frac{\pi}{2}-\alpha}). \tag{5.8}$$

We call α a uniform weak Aleksandrov constant for λ_ϵ.

Proof. Since μ and λ are weak Aleksandrov related discrete measures, Proposition 3.6 tells us that there exists an $\alpha' > 0$ such that for any closed set $\omega \subset S^{n-1}$,

$$\mu(\omega) \leq \lambda(\omega_{\frac{\pi}{2}-\alpha'}). \tag{5.9}$$

Choose ε' small enough so that $\mathrm{spt}\phi_{\varepsilon',j} \subset (u_j)_{\frac{\alpha'}{2}}$ for any j. If $u_j \in \omega_{\frac{\pi}{2}-\alpha'}$, then $u_j \in (v)_{\frac{\pi}{2}-\alpha'}$ for some $v \in \omega$. Thus, $(u_j)_{\frac{\alpha'}{2}} \subset (v)_{\frac{\pi}{2}-\frac{\alpha'}{2}}$. Combining everything, we conclude that for any $\varepsilon \leq \varepsilon'$, if $u_j \in \omega_{\frac{\pi}{2}-\alpha'}$, then

$$\mathrm{spt}\phi_{\varepsilon,j} \subset (u_j)_{\frac{\alpha'}{2}} \subset (v)_{\frac{\pi}{2}-\frac{\alpha'}{2}} \subset \omega_{\frac{\pi}{2}-\frac{\alpha'}{2}}. \tag{5.10}$$

Therefore, since all of the original mass of measure λ still remains in $\omega_{\frac{\pi}{2}-\frac{\alpha'}{2}}$ for λ_ε (with possible additional mass from the other u_j), we conclude that for any $\varepsilon \leq \varepsilon'$,

$$\lambda(\omega_{\frac{\pi}{2}-\alpha'}) \leq \lambda_\varepsilon(\omega_{\frac{\pi}{2}-\frac{\alpha'}{2}}). \tag{5.11}$$

Combining this with (5.9) and letting $\alpha := \frac{\alpha'}{2}$, we complete the proof. $\square$

Lemma 5.5. *Given a discrete measure μ not concentrated on a closed hemisphere and a discrete equal-weight λ, suppose they satisfy the weak Aleksandrov relation. Then, there* *exists an* $\varepsilon' > 0$ *such that for all* $0 < \varepsilon < \varepsilon'$, *there exists* $K_\varepsilon \in \mathcal{P}_\mu$ *solving* $\mu = \lambda_\varepsilon(K_\varepsilon, \cdot)$ *of the following form:*

$$K_\varepsilon := (\bigcap_{i=1}^{m} H^-(1/\beta_{\varepsilon,i}, v_i))^*, \tag{5.12}$$

where $\beta_{\varepsilon,i} = \rho_{K_\varepsilon}(v_i)$. *For each* $\varepsilon < \varepsilon'$, $R_{K_\varepsilon} = 1$, *there exists a uniform constant* $C_{\mu,\lambda}$ *such that:*

$$r_{K_\varepsilon} > C_{\mu,\lambda} > 0. \tag{5.13}$$

Remark. This constant depends on vectors v_i and the uniform weak Aleksandrov constant α from Lemma 5.4. △

Proof. Let ε' and α be constants given by Lemma 5.4. If the measure μ is not concentrated on a closed hemisphere, then K_ε and $C_{\mu,\lambda}$ exist by Lemma 9.3 in the Appendix. The independence of constant $C_{\mu,\lambda}$ is guaranteed by uniformity of α for all $\varepsilon' < \varepsilon$. □

For the rest of this section, we will work with constants ε', α, $C_{\mu,\lambda}$ and polytopes K_ε from previous Lemmas. Define $\mathrm{spt}_{i,j,\varepsilon} \subset S^{n-1}$ as the region of intersection between the support of $\phi_{\varepsilon,j}$ and $\boldsymbol{\alpha}_{K_\varepsilon}(v_i)$. We note two identities below. The first comes from the definition of λ_ε, and the second comes from $\mu = \lambda_\varepsilon(K_\varepsilon, \cdot)$.

$$\begin{aligned} \bar{\lambda}(\sum_{i=1}^{m} \mathrm{spt}_{i,j,\varepsilon}) &= \varepsilon \\ \bar{\lambda}(\sum_{j=1}^{k} \mathrm{spt}_{i,j,\varepsilon}) &= \varepsilon \mu_i. \end{aligned} \tag{5.14}$$

Before we proceed with the proof, we provide the general picture. Solving the Discrete Gauss Image problem corresponds to putting point-masses at u_j into corresponding normal cones v_i. By smoothing out these point masses, we get a sequence of bodies K_ε such that each original point mass has split into possibly several normal cones. The idea is that by making $\varepsilon \to 0$, the supports of λ_ε become smaller. This should force the supports to be fully contained in a normal cone eventually. If this happens for some body K, then $\mu = \lambda_\varepsilon(K, \cdot) = \lambda(K, \cdot)$. We will see that this does not always happen.

We will compute $\Phi(K_\varepsilon, \mu, \lambda_\varepsilon)$. Note that $\mathrm{spt}_{i,j,\varepsilon} \subset \boldsymbol{\alpha}_{K_\varepsilon}(v_i) \subset (v_i)_{\arccos C_{\mu,\lambda}}$, by definition and Lemma 3.5. Therefore, in particular, all logarithms are defined in the following expression:

$$\begin{aligned} \Phi(K_\varepsilon, \mu, \lambda_\varepsilon) &= \int \log \rho_{K_\varepsilon} d\mu + \int \log \rho_{K_\varepsilon^*} d\lambda_\varepsilon \\ &= \sum_{i=1}^{m} \mu_i \log \beta_{\varepsilon,i} + \sum_{i,j=1}^{i=m,j=k} \int_{\mathrm{spt}_{i,j,\varepsilon}} \log \frac{1}{\beta_{\varepsilon,i} v_i v} \frac{1}{\varepsilon} d\bar{\lambda}(v). \end{aligned} \tag{5.15}$$

Inside of the right integral, we can pull β_i out. Then summing over j and using above identities (5.14), we get that the first sum cancels out. We have

$$\Phi(K_\varepsilon, \mu, \lambda_\varepsilon) = -\sum_{i,j=1}^{i=m,j=k} \int_{\mathrm{spt}_{i,j,\varepsilon}} \log(v_i v) \frac{1}{\varepsilon} d\bar{\lambda}(v). \tag{5.16}$$

Lemma 5.6. *Under the conclusion of Lemma 5.5, there exists a subsequence of K_ε converging as $\varepsilon \to 0$ to some convex body $K \in \mathcal{P}_\mu$ of the form*

$$K = (\bigcap_{i=1}^{m} H^-(1/\alpha_i, v_i))^*, \tag{5.17}$$

where $\beta_{\varepsilon,i} \to \alpha_i > 0$ along this subsequence. For this body, $R_K = 1$ and $r_K \geq C_{\mu,\lambda}$.

Proof. Since for each $\varepsilon < \varepsilon'$, we have that $R_{K_\varepsilon} = 1$ and $r_{K_\varepsilon} > C_{\mu,\lambda}$, the standard compactness arguments apply.

□

Lemma 5.7. *For subsequence $K_\varepsilon \to K$ in Lemma 5.6,*

$$\Phi(K_\varepsilon, \mu, \lambda_\varepsilon) \to \Phi(K, \mu, \lambda). \tag{5.18}$$

Proof. We will assume the convergence for all sequences $K_\varepsilon \to K$ to simplify notation. This implies that $\rho_{K_\varepsilon} \to \rho_K$ and $\rho_{K_\varepsilon^*} \to \rho_{K^*}$. Therefore, $\int \log \rho_{K_\varepsilon} d\mu \to \int \log \rho_K d\mu$. Now for the second summand of Φ, consider the following:

$$\int \log \rho_{K_\varepsilon^*} d\lambda_\varepsilon - \int \log \rho_{K^*} d\lambda = \int \log \rho_{K_\varepsilon^*} - \log \rho_{K^*} d\lambda_\varepsilon + \int \log \rho_{K^*}(d\lambda_\varepsilon - d\lambda). \tag{5.19}$$

First, we note that $\rho_{K_\varepsilon^*} \geq 1$ and $\rho_{K^*} \geq 1$. Since for any a, b we have that $\log a - \log b \leq \frac{1}{min(a,b)}|a-b|$, we obtain $\log \rho_{K_\varepsilon^*} - \log \rho_{K^*} \leq |\rho_{K_\varepsilon^*} - \rho_{K^*}|$. Since $\rho_{K_\varepsilon^*} \to \rho_{K^*}$ converges uniformly and $\int d\lambda_\varepsilon = k$, we obtain that:

$$\int \log \rho_{K_\varepsilon^*} - \log \rho_{K^*} d\lambda_\varepsilon \to 0. \tag{5.20}$$

It is not hard to see that our bump function $\phi_{\varepsilon,i}$ works nicely for limits of continuous functions. That is, λ_ε converges to λ weakly as functionals on the space of continuous functions. This implies that the second summand of (5.19) approaches zero. Combining all of the above implies that $\Phi(K_\varepsilon, \mu, \lambda_\varepsilon) \to \Phi(K, \mu, \lambda)$.

□

Lemma 5.8. *For subsequence $K_\varepsilon \to K$ in Lemma 5.6, there exist a subsequence such that for all i, j*

$$, \int_{spt_{i,j,\varepsilon}} \frac{1}{\varepsilon} d\bar{\lambda} \to c_{i,j}. \tag{5.21}$$

Coefficients $c_{i,j}$ form a matrix $[c_{i,j}]$ with k columns and m rows such that each coefficient in the matrix is nonnegative, the sum over any column is equal to 1, and the sum over the i-th row is equal to μ_i.

Remark. This corresponds to how much of weight falls into normal cone of vertex. In the special case $\mu_i = 1$, for the equal-weights problem, $[c_{i,j}]$ is a doubly stochastic matrix. △

Proof. We start with the subsequence from Lemma 5.6. Since

$$0 \leq \int_{\mathrm{spt}_{i,j,\varepsilon}} \frac{1}{\varepsilon} d\bar{\lambda} \leq 1, \tag{5.22}$$

the standard compactness arguments insure convergence of volumes of supports for some further subsequence. The coefficients of the matrix are clearly nonnegative. Properties of the matrix follow from identities (5.14). □

From all the above, we obtain the following form for the functional $\Phi(K, \mu, \lambda)$.

Lemma 5.9. *For K obtained in Lemma 5.8, we have:*

$$\Phi(K, \mu, \lambda) = -\sum_{i,j=1}^{i=m,j=k} c_{i,j} \log(v_i u_j) > 0. \tag{5.23}$$

Note that if $v_i u_j \leq 0$ then $c_{i,j} = 0$, we force the term $c_{i,j} \log(v_i u_j)$ to be zero in the above sum.

Proof. Recall that from Lemma 5.6, we have a bound $r_{K_\varepsilon} > C_{\mu,\lambda}$ and $R_{K_\varepsilon} = 1$. Therefore, $\mathrm{spt}_{i,j,\varepsilon} \subset \boldsymbol{\alpha}_{K_\varepsilon}(v_i) \subset (v_i)_{\arccos C_{\mu,\lambda}}$, by Lemma 3.5. In particular, this means that $\log(v_i v)$ is continuous and bounded from below on $\mathrm{spt}_{i,j,\varepsilon} = \mathrm{spt}\phi_{\varepsilon,j} \cap \boldsymbol{\alpha}_{K_\varepsilon}(v_i)$, with bound independent of ε. This with Lemma 5.8 and the construction of the bump function implies that

$$\int_{\mathrm{spt}_{i,j,\varepsilon}} \log(v_i v) \frac{1}{\varepsilon} d\bar{\lambda}(v) \to c_{i,j} \log(v_i u_j). \tag{5.24}$$

Combining this with Lemma 5.7 and Equation (5.16), we obtain the stated result. □

The matrix $[c_{i,j}]$ is known as the solution matrix to transportation problems of transferring desirable material from m sources to k locations. All such matrices (k columns and m rows such that each coefficient in matrix is nonnegative, sum over any column is equal to 1, and sum over the i-th row is equal to μ_i) are known as transportation polytopes. For reference, see Section 8.1 in [15]. It is a convex polytope with known extreme points. The special case of this polytope, when $k = m$ and $\mu_i = 1$, is known as the Birkhoff-von Neumann theorem.

Lemma 5.10. *There is a natural one-to-one correspondence between vertices of transportation polytopes containing matrix $[c_{i,j}]$ and assignment functionals from $\mathbb{F}$.*

Proof. Corollary 8.1.4 in [15] provides a recursive construction for vertices of the transportation polytope. It is not hard to see from this construction that each vertex is an assignment functional, and every assignment functional is a vertex. □

Convexity of the transportation polytope implies the following:

Proposition 5.11. *There exist $0 \leq \theta_f \leq 1$ for $f \in \mathbb{F}$, such that $\sum_{f\in\mathbb{F}} \theta_f = 1$ and*

$$\Phi(K, \mu, \lambda) = -\sum_{f\in\mathbb{F}} \theta_f A(f). \tag{5.25}$$

Since $\Phi(K, \mu, \lambda) < \infty$, we obtain that there exists $f \in \mathbb{F}$ such that $u_j v_{f(j)} > 0$. In the above sum, if $A(f) = -\infty$, then $\theta_f = 0$.

Proof. From Lemma 5.9, $-\Phi(K, \mu, \lambda)$ is the element-wise product of matrices $[c_{i,j}]$ and $[\log(v_i u_j)]$. By convexity of the transportation polytope and Lemma 5.10, we obtain the above result. □

Now the Theorem 5.1 follows:

Theorem 5.1. First part. Unique maximizer. *Proof.* We have already seen that assignment functional can only be solution if it is maximizer, by Proposition 5.3. Suppose the assignment functional, $A(g)$, is maximized at exactly one $g \in \mathbb{F}$. Combining Proposition 5.11 and Proposition 5.2 we obtain that for the constructed body K in Lemma 5.8,

$$\Phi(K,\mu,\lambda) = -\sum_{f\in\mathbb{F}} \theta_f A(f) \geq -A(g) \geq \Phi(K,\mu,\lambda). \tag{5.26}$$

Therefore, $\theta_g = 1$, and thus $[c_{i,j}]$ is a vertex of the transport polytope. In particular, this means that from Lemma 5.8,

$$\begin{aligned} \int_{\mathrm{spt}_{i,j,\varepsilon}} \frac{1}{\varepsilon} d\bar{\lambda} \to 0 \text{ if } g(j) \neq i \\ \int_{\mathrm{spt}_{i,j,\varepsilon}} \frac{1}{\varepsilon} d\bar{\lambda} \to 1 \text{ if } g(j) = i. \end{aligned} \tag{5.27}$$

Consider some K_ε. We claim that if $\int_{\mathrm{spt}_{i,j,\varepsilon}} \frac{1}{\varepsilon} d\bar{\lambda} > \frac{1}{2}$, then $\lambda_j \in \overset{\circ}{\boldsymbol{\alpha}}_{K_\varepsilon}(v_i)$. This is because $\mathrm{spt}_{i,j,\varepsilon}$ is an intersection of the convex normal cone at a point v_i with a spherical disk of volume ε. If the intersection of such sets contains at least $\frac{1}{2}$ of the volume of disk, the center of the disk lies in the interior of the normal cone. So (5.27) implies that for small enough ε, $\lambda_j \in \overset{\circ}{\boldsymbol{\alpha}}_{K_\varepsilon}(v_i)$, when $g(j) = i$. Therefore, $\mu = \lambda(K_\varepsilon, \cdot)$. In particular, g is the solution function. $\square$

Remark. While we proved that for small enough ε over some subsequence K_ε is a solution, we can not guarantee that K is a solution. The problem is that vectors u_j may lie on the boundary of $\boldsymbol{\alpha}_K(v_i)$ in general (even if $\Phi(K,\mu,\lambda) = -A(g)$). $\triangle$

Proposition 5.12. *For any discrete λ and discrete μ not contained in a closed hemisphere, polytope solutions to the Discrete Gauss Image problem from the set $\mathcal{P}_\mu$:*

$$\left(\bigcap_{i=1}^{m} H^-(1/\alpha_i, v_i)\right)^* \tag{5.28}$$

form an open set in terms of $(\alpha_1, \ldots \alpha_m) \in \mathbb{R}^m$.

Proof. Polytope $K \in \mathcal{K}_o^n$ is a solution if and only if $u_j \in \boldsymbol{\alpha}_K(\overset{\circ}{v}_{f_K(j)})$. The latter allows for small perturbations of $(\alpha_1, \ldots \alpha_m)$. $\square$

Theorem 5.1. Second part. Nonunique maximizer. *Proof.* Given μ and λ, suppose there exists a solution K such that $\mu = \lambda(K, \cdot)$. Let P be a polytope solution guaranteed by Proposition 4.1. Note that the assignment function f with respect to the body P is a maximizer of $A(\cdot)$ by Proposition 5.3. Since $u_j \in \boldsymbol{\alpha}_P(\overset{\circ}{v}_{f(j)})$, for any small perturbation of u_j, P will still be a solution to the Gauss Image problem for small perturbations of λ, by Proposition 5.12. So, in particular, f will still be maximizer for small perturbations of u_j, again by Proposition 5.3. We will arrive at a contradiction by showing that if we have more than one maximizer, then we can always perturb u_j small enough so that f is no longer a maximizer of $A(\cdot)$.

If f, g are two maximizers of $A(\cdot)$, without loss of generality, suppose $f(1) \neq g(1)$. We have that

$$A(f) = \sum_{j=1}^{k} \log u_j v_{f(j)} = \sum_{j=1}^{k} \log u_j v_{g(j)} = A(g). \tag{5.29}$$

Since $f(1) \neq g(1)$, we can perturb u_1 sufficiently small to decrease $\log(\frac{u_1 v_{f(1)}}{u_1 v_{g(1)}})$, since $u_1 v_{f(1)} > 0$ and $u_1 v_{g(1)} > 0$. Thus, the above equality changes to inequality since all other terms are held constant. Therefore, for this perturbation, f is no longer a maximizer of $A(\cdot)$. ☐

6. Alterntative Approach

Since we are working with polytopes, one might wonder whether there exists a more algebraic approach that does not rely on the smoothing of measures. Indeed, such an approach exists. The advantage of this direction is a simple algebraic combinatorial system that does not rely on the smoothing argument. We will not pursue this approach in its full generality, as it does not yield a better result. Yet, we believe this technique is important to be stated.

Rockafellar first characterized the set of subdifferentials of convex functions. He proved that a set is cyclically monotone if and only if it is a subgradient of a convex funtion (see [39]). Later, a generalization of the cyclic monotonicity was introduced by Rochet and Rüschendorf (see [38, 41]), in the context of mathematical economics and mass transport. More recently, Arstein-Avidan, Sadovsky, and Wyczesany, in [4], generalized cyclic monotoic sets even further to what is called c-path-boundedess. The approach we will present is based on similar ideas.

Here, we consider the most simple case of the Discrete Gauss Image problem: same number of equal mass atoms for λ and μ. Let

$$\lambda = \sum_{j=1}^{m} \delta_{u_j} \text{ and } \mu = \sum_{j=1}^{m} \delta_{v_j}. \tag{6.1}$$

Lemma 6.1. *Suppose measures μ and λ are discrete, equal-weight, and weak Aleksandrov related. Suppose μ is not concentrated on a closed hemisphere. Then we can reorder indices so that the identity permutation maximizes $A(\cdot)$. In this case, $u_j v_j > 0$ for all j.*

Proof. This immediately follows from Lemma 4.3. ☐

For the rest of this section, we will fix the indexing so that the identity permutation maximizes $A(\cdot)$. This is guaranteed by Lemma 6.1. The next Lemma provides the algebraic system that is equivalent to the existence of the solution to the Gauss Image problem.

Lemma 6.2. *Suppose measures μ and λ are discrete, equal-weight, and weak Aleksandrov related. Suppose μ is not concentrated on a closed hemisphere. Consider any polytope $P \in \mathcal{P}_\mu$*

$$P = (\bigcap_{i=1}^{m} H^-(1/\alpha_i, v_i))^*, \tag{6.2}$$

where $\alpha_i > 0$. Consider the following system of equations

$$a_{j,i} < x_j - x_i \text{ for } i \neq j \in \{1 \ldots m\}, \tag{6.3}$$

where $a_{j,i} = \log \frac{u_j v_i}{u_j v_j}$ if $\frac{u_j v_i}{u_j v_j} > 0$, and $a_{i,j} = -\infty$ otherwise. We define $a_{i,i} = 0$ for convenience. Then $\mu = \lambda(P, \cdot)$ if and only if $x_i = \log(\alpha_i)$ solves this system of equations (6.3).

Proof. Fix P, and consider its dual P^*. Since $\frac{1}{\alpha_i v_i u_j}$ is the distance from the center to the intersection between a ray in the u_j direction and the hyperplane $H(1/\alpha_i, v_i)$, we obtain that:

$$\log \rho_{P^*}(u_j) = \min_{i=1}^{m} \log(\frac{1}{\alpha_i v_i u_j}). \tag{6.4}$$

In the last equation, we assume that $\log(\frac{1}{x}) = +\infty$ if $x < 0$. On the other hand, $u_j \in \alpha_P(\overset{\circ}{v_j})$ if and only if the ray in the u_j direction intersects $H(1/\alpha_j, v_j)$ strictly before it intersects any other $H(1/\alpha_i, v_i)$ for $j \neq i$. Therefore, $\mu = \lambda(P, \cdot)$ if and only if for all $j \neq i$, the following strict inequality holds:

$$\log(\frac{1}{\alpha_j v_j u_j}) < \log(\frac{1}{\alpha_i v_i u_j}), \tag{6.5}$$

again assuming that the right side is $+\infty$ if $\alpha_i v_i u_j < 0$. Recalling that $\alpha_j, \alpha_i, v_j u_j > 0$ and assuming that the log of a non-positive value is $-\infty$, the above is equivalent to:

$$\log \frac{u_j v_i}{u_j v_j} < \log \alpha_j - \log \alpha_i. \tag{6.6}$$

Defining x_k to be $\log \alpha_k$, we arrive at the conclusion. □

The conclusion to the following lemma is the analogue to the cyclic monotonicity condition. It gives us natural condition on the coefficients in the above lemma.

Lemma 6.3. *Suppose measures μ and λ are discrete, equal-weight, and weak Aleksandrov related. Suppose μ is not concentrated on a closed hemisphere. Suppose the identity permutation maximizes the assignment functional $A(\cdot)$. Then the maximizer is unique if and only if for any non-trivial permutation σ on $\{1 \dots m\}$,*

$$a_\sigma := \sum_{i=1}^{m} a_{i,\sigma(i)} < 0, \tag{6.7}$$

where coefficients $a_{i,j}$ are given by Lemma 6.2.

Proof. Let Id denote the identity permutation. Given any nontrivial permutation σ,

$$\begin{aligned} A(\sigma) < A(\mathrm{Id}) \Leftrightarrow \\ \sum_{j=1}^{m} \log(u_j v_{\sigma(j)}) < \sum_{j=1}^{m} \log(u_j v_j) \Leftrightarrow \\ a_\sigma = \sum_{j=1}^{m} a_{j,\sigma(j)} = \sum_{j=1}^{m} \log(\frac{u_j v_{\sigma(j)}}{u_j v_j}) < 0, \end{aligned} \tag{6.8}$$

again assuming that $\log x = -\infty$ if $x < 0$ or the expression inside is undefined.

□

We will show that the system established in Lemma 6.2 has a solution if the coefficients $a_{i,j}$ satisfy the condition of Lemma 6.3. This provides an alternative proof for the Discrete Equal-Weight Gauss Image problem. The system with the non-strict inequality already arose in the work of K. Wyczesany [47]. The proof is based on induction and Helly's theorem. We refer the reader to [47] for details of the proof.

Lemma 6.4 (K. Wyczesany, Lemma 4.2.8 in [47])**.** *Let $\alpha_{i,j} \geq -\infty$ for $i \neq j \in \{1 \dots m\}$, and let $a_{i,j} = 0$ for $i = j$. Then the following system of equations*

$$a_{j,i} \leq x_j - x_i \text{ for } i \neq j \in \{1 \dots m\} \tag{6.9}$$

has a real solution if and only if for any permutation σ,

$$\sum_{i=1}^{m} a_{i,\sigma(i)} \leq 0. \tag{6.10}$$

It is not hard to see that the above lemma provides us with a similar lemma for the strict system:

Lemma 6.5. *Let $\alpha_{i,j} \geq -\infty$ for $i \neq j \in \{1 \dots m\}$ and let $a_{i,j} = 0$ for $i = j$. Then the following system of equations:*

$$a_{j,i} < x_j - x_i \text{ for } i \neq j \in \{1 \dots m\} \tag{6.11}$$

has a real solution if and only if for any non trivial permutation σ,

$$a_\sigma := \sum_{i=1}^{m} a_{i,\sigma(i)} < 0. \tag{6.12}$$

Remark. One can prove this directly, with a similar approach to the proof Lemma 6.4. △

Proof. Assume the system of equations has a solution. Then for any non-identical permutation σ,

$$\begin{aligned} a_\sigma = \sum_{i=1}^{m} a_{i,\sigma(i)} = \sum_{\sigma(i)\neq i}^{m} a_{i,\sigma(i)} + \sum_{\sigma(i)=i}^{m} a_{i,i} = \\ \sum_{\sigma(i)\neq i}^{m} a_{i,\sigma(i)} < \sum_{\sigma(i)\neq i}^{m} (x_i - x_{\sigma(i)}) = 0, \end{aligned} \tag{6.13}$$

where the last equality follows by simply opening the sum and canceling terms.

For the other direction, assume the conditions on coefficients $\alpha_{i,j}$. Then for any σ not equal to the identity permutation,

$$\sum_{i=1}^{m} a_{i,\sigma(i)} < 0. \tag{6.14}$$

Because there are finitely many permutations, we can choose uniform ε such that for each σ not equal to the identity permutation,

$$\sum_{i=1}^{m} a_{i,\sigma(i)} + \varepsilon < 0. \tag{6.15}$$

Choose new coefficients $\bar{a}_{i,j} = a_{i,j} + \frac{\varepsilon}{m}$ if $i \neq j$, and $\bar{a}_{i,j} = a_{i,j} = 0$ otherwise. These coefficients satisfy the condition of Lemma 6.4. Hence, there is a solution to the following system of equations:

$$\bar{a}_{j,i} \leq x_j - x_i \text{ for } i \neq j \in \{1 \dots m\}, \tag{6.16}$$

since $\alpha_{i,j} < \bar{\alpha}_{i,j}$ for $i \neq j$. The same solution solves the original system. □

Now we give alternative proof of the discrete equal-weight problem.

Theorem 6.6. *Suppose μ and λ are discrete equal-weight measures that satisfy the weak Aleksandrov inequality. Suppose μ is not concentrated on a closed hemisphere. Then $A(\cdot)$ is maximized by exactly one $\sigma \in \mathbb{F}$ if and only if there exists a polytope P with vertices $r_P(v_i)$ solving $\mu = \lambda(P, \cdot)$.*

Proof. By Lemma 6.1, one can order the coefficients so that the identity permutation maximizes the functional $A(\cdot)$. Then by Lemma 6.2, there exists a P solving $\mu = \lambda(P, \cdot)$ if and only if there is a solution to the system of equations (6.3). By Lemma 6.5, the system of equations (6.3) has a solution if and only if condition (6.12) is satisfied for $\alpha_{i,j}$. The condition (6.12) is equivalent to the unique maximization of the assignment functional, by Lemma 6.3. □

7. The Assignment Problem from Geometric Point of View

As was shown by Theorem 5.1 the question of the existence of the solution to the Discrete Gauss Image problem is equivalent to the uniqueness of the maximizer for the assignment functional. Let us now analyze this question. In this section, we will provide a geometric condition that insures that the maximizer is unique, and, hence, there exists a solution to the Discrete Gauss Image Problem. This condition is equivalent to the statement that if $\sigma_1 \neq \sigma_2 \in \mathbb{F}_{\mu,\lambda}$ then $A(\sigma_1) = A(\sigma_2)$, which forces $A(\cdot)$ to be uniquely maximized. This can be seen to be not far away from necessary, yet for the true necessary condition, one has to restrict the class of permutations $\mathbb{F}_{\mu,\lambda}$ to consider. In the next section, we are going to analyze the uniqueness of maximizer from the generic point of few.

It turns out that the question about the uniqueness of maximizers of the assignment functional for the dimension $n = 2$ is very easy to understand. Consider the following example. Let P be any regular convex polytope with the center in its interior. Choose set $\{v_1 \dots v_m\}$ of distinct unit vectors such that $r_P(v_i)$ are all vertices of P. Since $n = 2$ we can also ensure that vector subscript notation is clock-wise ordered. Now let the set $\{u_1 \dots u_m\}$ be normals to the facets of P such that $u_j \perp [r_P(v_j), r_P(v_{j+1})]$ for $j < m$ and $u_m \perp [r_P(v_m), r_P(v_1)]$. Let

$$\lambda = \sum_{j=1}^{m} \delta_{u_j} \text{ and } \mu = \sum_{j=1}^{m} \delta_{v_j}. \tag{7.1}$$

These two measures are weak Aleksandrov related and μ is not concentrated on a closed hemisphere, hence satisfying conditions of Theorem 5.1. We will now prove that $A(\cdot)$ is maximized at exactly two permutations from $\mathbb{F}_{\mu,\lambda}$: the identity permutation σ_{Id} and a cycle $\sigma_s = (1 \dots m)$. First, a similar argument to Proposition 3.4 shows that μ and λ are weak Aleksandrov related. Clearly, μ is not concentrated on a closed hemisphere. By a simple computation of $\Phi(P, \mu, \lambda)$ analogues to derivation of Proposition 5.2 and equations (5.4) and (5.5) one can show that for any permutation $\sigma \neq \sigma_s$ and $\sigma \neq \sigma_{Id}$:

$$-A(\sigma_{id}) = -A(\sigma_s) = \Phi(P, \mu, \lambda) < -A(\sigma) \tag{7.2}$$

Combining all these Theorem 5.1 gives us that there doesn't exist a solution $K \in \mathcal{K}_o^n$ such that $\mu = \lambda(K, \cdot)$. It is interesting to note that the solution K_ε for λ_ε,μ-problem from Lemma

5.6 converge to P, so in particular, the method established in Existence and Uniqueness section of this work identifies the body P from which measures were constructed.

This example is the essence of the Assignment Problem. In dimension two if one starts with discrete equal-weight μ and λ satisfying weak Aleksandrov condition such that μ is not concentrated on a closed hemisphere we either obtain that there exist polytope $P \in \mathcal{P}_\mu$ such that $\mu = \lambda(P, \cdot)$ or there was a polytope $P \in \mathcal{P}_\mu$ such that $\{u_1 \dots u_m\}$ where normals to its facets. The answer depends on the uniqueness of the maximizer. Moreover, the failure of uniqueness precisely determines $P \in \mathcal{P}_\mu$ (up to scaling) and the assignment of u_j to specific side of P in the sense of $u_j \perp [r_P(v_i), r_P(v_{i+1})]$.

Now, let's analyze the higher dimensional picture while additionally dropping the equal-weight assumption for μ. First, we define a geometric condition which is related to the uniqueness of the maximizer. This condition comes from "lifting" the mentioned two-dimensional example into higher dimensions.

Definition 7.1. Given set of distinct vectors $\{u_1 \dots u_l\}$ and distinct vectors $\{v_1 \dots v_l\}$ with $l \geq 2$ suppose there exist a piecewise linear closed curve γ with vertices $\{x_1 \dots x_l\}$ such that

- $x_i = \lambda_i v_i$ for some $\lambda_i > 0$
- $u_i \perp [x_i, x_{i+1}]$ for $1 \leq i \leq l-1$
- $u_l \perp [x_l, x_1]$

The curve γ is called an edge-normal loop of two sets.

Remark. This curve doesn't always exist. If it exists it is unique up to dilation.

△

Clearly, any piecewise linear closed curve not passing through the center provides two sets $\{u_1 \dots u_l\}$ and $\{v_1 \dots v_l\}$ to which it is an edge-normal loop. Going back to the example discussed at the beginning of this section, any two dimensional polytope defines an edge-normal loop. The next two propositions establish the relation between the edge-normal loops and the values of the assignment functional for different permutations.

Proposition 7.2. *Suppose a piecewise linear closed curve γ with verticies $\{x_1 \dots x_l\}$ is an edge normal loop of $\{u_1 \dots u_l\}$ and $\{v_1 \dots v_l\}$. Then*

$$A(\sigma_{id}) = A(\sigma_s) \tag{7.3}$$

where σ_s is cycle permutation $(1 \dots l)$.

Proof. Let $u_{l+1} = u_1$, $v_{l+1} = v_1$, $x_{l+1} = x_1$ and $\lambda_{l+1} = \lambda_1$ then by defenition:

$$A(\sigma_{id}) = A(\sigma_s) \Leftrightarrow$$

$$\sum_{j=1}^{l} \log(u_j v_j) = \sum_{j=1}^{l} \log(u_j v_{j+1}) \tag{7.4}$$

If $0 \in \gamma$ then $0 \in [x_i, x_{i+1}] = [\lambda_i v_i, \lambda_{i+1} v_{i+1}]$ and hence $u_i v_i = 0$ and $u_i v_{i+1} = 0$ which forces $A(\sigma_{id}) = A(\sigma_s) = -\infty$ by convention and establishes the above equality. Suppose $0 \notin \gamma$, then for all j, $u_j v_j > 0$ and $u_j v_{j+1} > 0$. Hence the above equality is equivalent to:

$$\prod_{j=1}^{l} \frac{u_j v_j}{u_j v_{j+1}} = 1 \tag{7.5}$$

Now $u_i \perp [x_i, x_{i+1}]$ is exactly $u_i(x_i - x_{i+1}) = 0$ which is equivalent to $\frac{\lambda_{i+1}}{\lambda_i} = \frac{u_i v_i}{u_i v_{i+1}}$. Therefore, the above is equivalent to:

$$\prod_{j=1}^{l} \frac{\lambda_{i+1}}{\lambda_i} = 1 \tag{7.6}$$

which holds since $\lambda_{l+1} = \lambda_1$.

□

We now show the reverse statement.

Proposition 7.3. *Suppose for sets* $\{u_1 \dots u_l\}$ *and* $\{v_1 \dots v_l\}$

$$A(\sigma_{id}) = A(\sigma_s) > -\infty \tag{7.7}$$

Then there exist an edge-normal loop of $\{u_1 \dots u_l\}$ *and* $\{v_1 \dots v_l\}$.

Proof. Similar to previous proposition, since $A(\sigma_{id}) = A(\sigma_s) > -\infty$ we obtain

$$\prod_{j=1}^{l} \frac{u_j v_j}{u_j v_{j+1}} = 1 \tag{7.8}$$

Let $\lambda_1 = 1$. Then recursively define λ_i for $i \leq l$ using the relation $\frac{\lambda_{i+1}}{\lambda_i} = \frac{u_i v_i}{u_i v_{i+1}}$. By recursive defenition we obtain that for all $i < l$, $u_i \perp [x_i, x_{i+1}]$. Then using (7.8) we obtain:

$$\frac{\lambda_l}{\lambda_1} = \prod_{j=1}^{l-1} \frac{\lambda_{i+1}}{\lambda_i} = \prod_{j=1}^{l-1} \frac{u_j v_j}{u_j v_{j+1}} = \frac{u_l v_1}{u_l v_l} \tag{7.9}$$

Therefore we also obtain $u_l \perp [x_l, x_1]$. □

Definition 7.4. Discrete equal-weight measures μ and λ are called edge-normal loop free if for any $\sigma, \sigma' \in S_m$ and given any l such that $2 \leq l \leq m$, there dose not exist and edge normal loop for $\{u_{\sigma(1)} \dots u_{\sigma(l)}\}$ and $\{v_{\sigma'(1)} \dots v_{\sigma'(l)}\}$

Before addressing non-equal weight measure μ let us first illustrate the relation between edge-normal loop free condition and the uniqueness of the assignment functional.

Theorem 7.5. *Suppose μ and λ are discrete equal-weight measures which are weak Aleksandrov related and such that μ is not concentrated on a closed hemisphere. Suppose μ and λ are edge-normal loop free. Then there exist a unique $\sigma \in \mathbb{F}_{\mu,\lambda}$ which maximizes $A(\cdot)$.*

Proof. Since μ and λ are weak Aleksandrov related by Lemma 4.3 the set $\mathbb{F}_{p,\mu,\lambda}$ is nonempty. Hence, there exist a $\sigma \in \mathbb{F}_{p,\mu,\lambda}$ such that $A(\sigma) > -\infty$. By reordering indices we can assume that the maximizers is identity permutation. Now chose any other permutation $\sigma \in F_{p,\mu,\lambda}$. Let $\tau_1 \dots \tau_k = \sigma$ be its decomposition into non-trivial non-intersecting cycles.

Let $\tau_1 = (j_1 \dots j_v)$. Consider ordered subsets $\{u_{j_1}, \dots u_{j_v}\}$ and $\{v_{j_1} \dots v_{j_v}\}$. Since μ and λ are edge-normal loop free from, we obtain from Proposition 7.3 that $A(\tau_1') \neq A(\sigma_{id}^1)$ where σ_{id}^1 is identity permutation on the set $\{j_1 \dots j_v\}$ and τ_1' is a restriction of τ_1 to set $\{j_1 \dots j_v\}$. If $A(\tau_1') > A(\sigma_{Id}^1)$ then $A(\tau_1) > A(\sigma_{Id})$. Thus, the identity permutation is not a maximizer. Contradiction. Hence, $A(\tau_1') < A(\sigma_{Id}^1)$. Similarly, we obtain for all $1 \leq i \leq k$ that $A(\tau_i') < A(\sigma_{Id}^i)$. Therefore, $A(\sigma) < A(\sigma_{Id})$. Since σ was an arbitrary permutation we are done. □

Combining this with Theorem 5.1 we obtain:

Corollary 7.6. *Suppose μ and λ are discrete equal-weight measures which are weak Aleksandrov related and such that μ is not concentrated on a closed hemisphere. Suppose μ and λ are edge-normal loop free. Then, there exists a polytope $P \in \mathcal{P}_\mu$ such that $\lambda(P, \cdot) = \mu$.*

We now show the reverse statement which gives another insight into the geometry behind edge-normal loop condition.

Proposition 7.7. *Pick any μ. Given polytope $P \in \mathcal{P}_\mu$, consider a closed path γ of its adjacent vertices using the edges of P. Suppose $\{x_1 \dots x_k\}$ are its vertices in order. Let $x_{k+1} = x_k$. Pick u_i such that $[x_i, x_{i+1}] \in H_P(u_i)$. Then γ is an edge normal loop for sets $\{v_1 \dots v_k\}$ and $\{u_1 \dots u_k\}$.*

Proof. Immediate. □

In particular, this shows that in some sense all "relevant" edge-normal loops come from 1-skeleton of convex polytopes. If the edge-normal loop is not realizable as a part of 1-skeleton of a convex polytope with the center in its interior it doesn't affect the solution of the Gauss Image Problem. This also shows that in general normals to edges are quite rigid if there exists an edge-normal loop of them. To formulate this more precisely, consider the following question: Suppose we start with some discrete equal-weight measure μ and polytope $P \in \mathcal{P}_\mu$. Suppose λ is a discrete-equal weight measure. Now let us assume that P almost solves the Gauss Image Problem with σ_{Id} assignment, but it happened that some of the vectors u_i are actually on the boundary of the normals cones $u_j \in \boldsymbol{\alpha}_P(v_j)$. Can we change the polytope a bit, moving its vertices along the rays v_j to obtain $u_j \in \mathring{\boldsymbol{\alpha}}_P(v_j)$, so that everything falls inside of normal cones? If we have only one vector on the boundary, then there is no problem, we can move the corresponding vertex a bit outside to increase the normal cone. What about the general case? Well, we can do this if and only if there are no edge-normal loops.

We now turn to the statement for non-equal weight measure μ. We will need some basic combinatoric machinery established before we proceed. Unfortunately, we were not able to find a good reference for these results. Given permutation σ we can uniquely decompose it into disjoint cycles $\tau_j = (x^j_1 \dots x^j_{s_j})$ where by s_j we denote the length of the cycle. We prove a similar statement for assignment functionals:

Proposition 7.8. *Let λ be a discrete equal-weight measure and μ be a discrete measure. Suppose $f, g \in \mathbb{F}_{\mu,\lambda}$. Then, there exist a permutation $\sigma \in S_k$ such that $f \circ \sigma = g$. Moreover, there exist a product of disjoint cycles $\tau_1, \dots, \tau_l \in S_k$, such that for each $\tau_j = (x^j_1 \dots x^j_{s_j})$ function f is injective with restrict to the set $\{x^j_1, \dots, x^j_{s_j}\}$ and $f \circ \tau_1 \dots \tau_l = g$.*

Proof. Since $|f^{-1}(i)| = |g^{-1}(i)|$ given any $i \in 1 \dots m$ choose a bijective function h_i from set $g^{-1}(i)$ into set $f^{-1}(i)$. Since all functions h_i are bijections with non-intersecting supports and images we can define $\sigma \in S^{n-1}$ to be permutations satisfying $\sigma(j) = h_{g(j)}(j)$. Then, $f \circ \sigma(j) = f h_{g(j)}(j) = f(f^{-1}(g(j))) = g(j)$, which ensures the first claim.

Now pick any σ, satisfying $f \circ \sigma = g$. Decompose σ into a product of disjoint cycles $\phi_1 \dots \phi_v$. Suppose, without loss of generality, cycle $\phi_1 = (x_1 \dots x_{s(j)})$ does not satisfy the claim. We will show that we can, then, always split it into two more cycles ω_1 and ω_2 of strictly smaller size such that $f \circ \omega_1 \omega_2 \phi_2 \dots \phi_v = g$ and all cycles are disjoint. Then, since, in general, we can only do finitely many splitting, eventually we will have that f is injective with respect to any cycle in the decomposition.

We are only left to prove that if f is not injective with respect to elements permuted by ϕ_1, then we can split ϕ_1 into ω_1 and ω_2, so that $f \circ \phi_1 = f \circ \omega_1\omega_2$. Given $\phi_1 = (x_1 \dots x_{s(1)})$ let i, j be the indices such that $f(x_i) = f(x_j)$. Without loss of generality, assume $i = 1$. Then we can define $\omega_1 = (x_1 \dots x_{j-1})$ and $\omega_2 = (x_j \dots x_{s(1)})$. Now, ϕ_1 is equal to $\omega_1\omega_2$ everywhere besides x_{j-1} and $x_{s(1)}$. Yet, $f \circ \phi_1(x_j) = f(x_j) = f(x_1) = f \circ \omega_1(x_{j-1})$ and $f \circ \phi_1(x_{s(1)}) = f(x_1) = f(x_j) = f \circ \omega_2(x_{s(1)})$. Therefore, $f \circ \phi_1 = f \circ \omega_1\omega_2$. □

Now with the combinatorial result in hand, we similarly obtain the previous results for a bigger class of measures.

Definition 7.9. Discrete equal-weight measures λ and discrete measure μ are called edge-normal loop free if given any l such that $2 \leq l \leq m$ and given any $\sigma \in S_k$ and $\sigma' \in S_m$ and given any l such that $2 \leq l \leq m$, there dose not exist and edge normal loop for $\{u_{\sigma(1)} \dots u_{\sigma(l)}\}$ and $\{v_{\sigma'(1)} \dots v_{\sigma'(l)}\}$.

Theorem 7.10. *Let λ be a discrete equal-weight measure and μ be a discrete measure. Suppose they are weak Aleksandrov related, edge-normal loop free and μ is not concentrated on a closed hemisphere. Then there exist a unique $f \in \mathbb{F}_{\mu,\lambda}$ which maximizes $A(\cdot)$.*

Proof. Since μ and λ are weak Aleksandrov related by Theorem 5.1 the set $\mathbb{F}_{p,\mu,\lambda}$ is nonempty. Hence, there exist a $f \in \mathbb{F}_{p,\mu,\lambda}$ such that $A(f) > -\infty$.. Suppose $f, g \in \mathbb{F}_{p,\mu,\lambda}$ are maximizers of $A(\cdot)$. By Proposition 7.8 there exist disjoint cycles $\tau_1, \dots, \tau_l \in S_k$, such that for each $\tau_j = (x_1^j \dots x_{s_j}^j)$ function f is injective with restrict to the set $\{x_1^j, \dots, x_{s_j}^j\}$ and $f \circ \tau_1 \dots \tau_l = g$. Note that since f is injective on any cycle, $s_j \leq m$ for any j.

Let f_i and g_i for $1 \leq i \leq l$ be the restrictions of f and g to the subset $\{x_1^j, \dots, x_{s_j}^j\}$. Then,

$$A(f) = \sum_{i=1}^{l} A(f_i)$$

$$A(g) = \sum_{i=1}^{l} A(g_i) = \sum_{i=1}^{l} A(f_i \circ \tau_i)$$

Notice that $A(f_i \circ \tau_i) = A(g_i)$. If, it happened that for some i, $A(f_i \circ \tau_i) > A(f_i)$, then $A(f \circ \tau_i) > A(f)$, and hence f is not the maximizer. Therefore, $A(f_i \circ \tau_i) \leq A(f_i)$. Since μ and λ are edge-normal loop free, sets $\{u_{x_1^j} \dots u_{x_{s_j}^j}\}$ are $\{v_{x_1^j} \dots v_{x_{s_j}^j}\}$ are edge-normal loop free, and, hence, $A(f_i \circ \tau_i) \neq A(f_i)$ for all i. Therefore, $A(f_i \circ \tau_i) < A(f_i)$. Thus, $A(g) < A(f)$. Contradiction. □

Combining this with Theorem 5.1 we obtain:

Corollary 7.11. *Let λ be a discrete equal-weight measure and μ be a discrete measure. Suppose they are weak Aleksandrov related, edge-normal loop free and μ is not concentrated on a closed hemisphere. Then, there exists a polytope $P \in \mathcal{P}_\mu$ such that $\lambda(P, \cdot) = \mu$.*

Proof. The proof is an immediate combination of Theorem 5.1 and Theorem 7.10 □

8. The Assignment Problem from Generic Point of View

We will now investigate The Assignment Problem from generic point of view. First, we will show that the maximizer is unique in the generic sense. Suppose we are given a discrete, equal-weight measure λ and a discrete measure μ, such that $\mu(S^{n-1}) = \lambda(S^{n-1})$. We define

$\mathcal{A}$ to be the class of all possible pairs of measures μ', λ' with variations of directions of the point masses of μ and λ. More formally, we can start with some $m \in \mathbb{N}$ and coefficients $\mu_i \in \mathbb{N}$ and consider all possible pairs of measures (μ and λ). We will additionally impose that μ is a discrete measure with fixed coefficients and λ is a discrete, equal-weight measure with fixed $k = \sum_i^m \mu_i$. We can parameterize this class as product of $m + k$ spheres, with a sphere for each vector. That is, each $(\mu', \lambda') \in \mathcal{A}$ has representation $(v'_1, \ldots, v'_m, u'_1, \ldots, u'_k)$. The set $\mathcal{A}$ naturally inherits the topology from spheres. In $\mathcal{A}$, we also require that u_j be distinct and v_i be distinct. In terms of parameterization, this constitutes an open subset of the product of spheres.

Definition 8.1. Space $\mathcal{A}$ constructed above is called the μ problem space. Sometimes, we write $\mathcal{A}_\mu$ to emphasize the original measure from its construction. Note that it only depends on the dimension n, constant $k \in \mathbb{N}$, and coefficients $\mu_i \in \mathbb{N}$ for $1 \leq i \leq k$.

Proposition 8.2. *Pairs of measures $(\mu, \lambda) \in \mathcal{A}$ satisfying the weak Aleksandrov inequality form an open set in the inherited topology from the product of spheres.*

Proof. Take some $(v_1, \ldots, v_m, u_1, \ldots, u_k) \sim (\mu, \lambda) \in \mathcal{A}$ satisfying the weak Aleksandrov relation. Then by Proposition 3.6, there exists a uniform $\alpha > 0$ such that for each closed set $\omega \subset S^{n-1}$:

$$\mu(\omega) \leq \lambda(\omega_{\frac{\pi}{2}-\alpha}). \tag{8.1}$$

Let $(v'_1, \ldots, v'_m, u'_1, \ldots, u'_k) \sim (\mu', \lambda') \in \mathcal{A}$ be any pair of measures satisfying $u_j u'_j > \cos(\frac{\alpha}{4})$ and $v_i v'_i > \cos(\frac{\alpha}{4})$. Note that all such possible measures form an open ball around (μ, λ) in the product topology. Then for any compact, convex set $\omega \subset S^{n-1}$, if $v'_i \in \omega$, then $v_i \in \omega_{\frac{\alpha}{4}}$. Hence,

$$\mu'(\omega) = \mu'(\omega \cap \{v'_1, \ldots, v'_m\}) \leq \mu(\omega_{\frac{\alpha}{4}} \cap \{v_1, \ldots, v_m\}). \tag{8.2}$$

We can apply Proposition 3.6 to $\omega_{\frac{\alpha}{4}} \cap \{v_1, \ldots, v_m\}$ to obtain:

$$\mu'(\omega) \leq \mu(\omega_{\frac{\alpha}{4}} \cap \{v_1, \ldots, v_m\}) \leq \lambda((\omega_{\frac{\alpha}{4}} \cap \{v_1, \ldots, v_m\})_{\frac{\pi}{2}-\alpha}). \tag{8.3}$$

Clearly, $(\omega_{\frac{\alpha}{4}} \cap \{v_1, \ldots, v_m\})_{\frac{\pi}{2}-\alpha} \subset \omega_{\frac{\pi}{2}-\frac{3}{4}\alpha}$. Thus,

$$\mu'(\omega) \leq \lambda(\omega_{\frac{\pi}{2}-\frac{3}{4}\alpha}). \tag{8.4}$$

Now if $u_j \in \omega_{\frac{\pi}{2}-\frac{3}{4}\alpha}$, then $u'_j \in \omega_{\frac{\pi}{2}-\frac{2}{4}\alpha}$. We obtain:

$$\mu'(\omega) \leq \lambda'(\omega_{\frac{\pi}{2}-\frac{2}{4}\alpha}). \tag{8.5}$$

Therefore, μ', λ' are weak Aleksandrov related. $\square$

Proposition 8.3. *Pairs of measures $(\mu, \lambda) \in \mathcal{A}$ such that μ is not concentrated on a closed hemisphere form an open set.*

Proof. Consider a sequence of $(\mu^n, \lambda^n) \to (\mu, \lambda)$, where μ^n are concentrated on a closed hemisphere. For each n, there exists a $u_n \in S^{n-1}$ such that for all i, $u_n v_i^n \leq 0$. By compactness, there exists a subsequence such that $u_n \to u$. Since $v_i^n \to v_i$, then along the subsequence $u_n v_i^n \to u v_i$. Hence, for all i, $u v_i \leq 0$. Thus, μ_n is concentrated on a closed hemisphere. Therefore, the set (μ, λ) where μ is concentrated on a closed hemisphere is closed, and its complement is an open set. □

Theorem 8.4. *Given $\mathcal{A}$, the pairs of measures $(\mu, \lambda) \in \mathcal{A}$ for which there exists a polytope solution to the Discrete Gauss Image problem form a dense open set in an open set of measures (μ, λ) satisfying the weak Aleksandrov relation, where μ is not concentrated on a closed hemisphere.*

Proof. Let $\mathcal{A}_c$ be subset of $\mathcal{A}$ consisting of pairs of measures that are weak Aleksandrov related and where μ is not concentrated on a closed hemisphere. Openess of $\mathcal{A}_c$ follows from Propositions 8.3 and 8.2. Define $A'(f) := e^{A(f)}$, where $A'(f) = 0$ if $A(f) = -\infty$. We note that the set $\mathbb{F}$ of assignment functions remains the same for any pair of measures from $\mathcal{A}$. For each $f \in \mathbb{F}$, we also define the set $\mathcal{A}_f := \{(\mu, \lambda) \mid \forall j \; u_j v_{f(j)} > 0\}$.

First, we note that $A'(f)$ is a continuous function on $\mathcal{A}$ as a function from pairs of measures into $\mathbb{R}$. Moreover, $A'(f) = A'(g)$ if and only if $A(f) = A(g)$, and $A'(f) > A'(g)$ if and only if $A(f) > A(g)$. Therefore, $A(\cdot)$ is uniquely maximized for a fixed pair of measures if and only if $A'(\cdot)$ is uniquely maximized for the same pair. Recall that by Theorem 5.1, to prove the statement of the theorem, it suffices to show that the assignment functional is uniquely maximized on a dense open subset of $\mathcal{A}_c$.

Suppose for some pair $(\mu, \lambda) \in \mathcal{A}_c$, the functional is uniquely maximized. Thus, there exists $f \in \mathbb{F}$, such that $A(f) > A(g)$ for every other $g \in \mathbb{F}$. Moreover, $A(f) > -\infty$ by Theorem 5.1, since $(\mu, \lambda) \in \mathcal{A}_c$. Therefore, $A'(f) > A'(g)$ for every other $g \in \mathbb{F}$, and $A'(f) > 0$. Then, since there are finitely many $g \in \mathbb{F}$, by continuity of $A'(f)$ and $A'(g)$ with respect to pairs of measures, there exists a neighborhood of (μ, λ) such that $A'(f)$ is still a maximizer and $A'(f) > 0$. Thus, $A(f)$ is a unique maximizer for a neighborhood of (μ, λ). This shows that the set where the assignment functional is uniquely maximized is open in $\mathcal{A}$. Since $\mathcal{A}_c$ is open, this set is also open in $\mathcal{A}_c$.

Consider some $\mathcal{A}_f \cap \mathcal{A}_g = \varnothing$. We would like to show that $A(f) \neq A(g)$ on a dense open subset of $\mathcal{A}_f \cap \mathcal{A}_g$. First of all, $A(f)$ and $A(g)$ are continuous functions on $\mathcal{A}_f \cap \mathcal{A}_g$, and thus $A(f) \neq A(g)$ on an open subset of $\mathcal{A}_f \cap \mathcal{A}_g$. Suppose now for some pair $(\mu, \lambda) \in \mathcal{A}_f \cap \mathcal{A}_g$, we have $A(f) = A(g)$. Then:

$$\begin{aligned} A(f) = \sum_{j=1}^{k} \log u_j v_{f(j)} = \sum_{j=1}^{k} \log u_j v_{g(j)} = A(g) \Leftrightarrow \\ \sum_{j=1}^{k} \log\left(\frac{u_j v_{f(j)}}{u_j v_{g(j)}}\right) = 0. \end{aligned} \tag{8.6}$$

Since $f \neq g$, we can find a j such that $f(j) \neq g(j)$. Since the v_i are distinct, we have that $v_{f(j)} \neq v_{g(j)}$. Thus, there exists a small variation of u_j such that all other terms are held constant while $\log(\frac{u_j v_{f(j)}}{u_j v_{g(j)}})$ changes, which makes $A(f) \neq A(g)$. This shows that $A(f) \neq A(g)$ is a dense subset of $\mathcal{A}_f \cap \mathcal{A}_g$, making it a dense open subset of $\mathcal{A}_f \cap \mathcal{A}_g$.

Now consider some $(\mu, \lambda) \in \mathcal{A}_c$ such that the functional is not uniquely maximized. Suppose $\{f_i\}_{i \in I}$ are all proper assignment functions. That is, $A(f_i) > -\infty$. Let $\mathcal{A}_I := \bigcap \mathcal{A}_{f_i}$, which is non-empty open set containing (μ, λ). From the previous argument, for each pair $i_1, i_2 \in I$, $A(f_{i_1}) \neq A(f_{i_2})$ on a dense open subset of $\mathcal{A}_{f_{i_1}} \cap \mathcal{A}_{f_{i_2}}$. Hence, $A(f_{i_1}) \neq A(f_{i_2})$ on a dense open subset of $\mathcal{A}_I$. Since a finite intersection of dense open sets is dense and open, there exists a dense open set $\mathcal{B}_1 \subset A_I$ such that all $A(f_i)$ are distinct. Hence, all $A'(f_i)$ are distinct on $\mathcal{B}_1$.

Since $(\mu, \lambda) \in \mathcal{A}_c$, by Theorem 5.1, there exists an $f \in \mathbb{F}$ such that $A'(f) > 0$. By continuity, there exists a neighborhood $\mathcal{B}_2$ of (μ, λ) such that $A'(f) > A'(g)$ for each $g \notin \{f_i\}_{i \in I}$, since $A'(g) = 0$ at (μ, λ). Combining both sets $\mathcal{B}_1$ and $\mathcal{B}_1$, we see that for each $(\mu, \lambda) \in \mathcal{B} := \mathcal{B}_1 \cap \mathcal{B}_2$, the functional is uniquely maximized. Moreover, $\mathcal{B}$ is a dense open set of some neighborhood of (μ, λ). Therefore, there exists a pair of measures with a unique maximizer sufficiently close to (μ, λ). This shows that the set where assignment functional is uniquely maximized is dense in $\mathcal{A}_c$.

□

Remark. Instead of the above generic formulation, we can write generic property with the respect to the Zariski topology. Since for each $f, g \in \mathbb{F}$, the condition

$$A(f) - A(g) = 0 \tag{8.7}$$

are zeros of an algebraic function, it suffices to show that the set

$$\{(\mu, \lambda) \mid \forall f, g \ A(f) \neq A(g)\} \tag{8.8}$$

is nonempty. We will still need to separately require the non-concentration condition on μ and the weak Aleksandrov condition. △

Finally, we will give some examples of pairs of measures for which there does not exist a unique maximizer.

Example 8.5. Let $n = 3$ and choose some constant $l > 3$. Pick any small regular spherical polygon with $2l$ vertices contained in a closed hemisphere. (That is, a polygon on sphere with edges being great-circle arcs, and same angles between planes defined by the consecutive great-circle arcs.) Iteratively label vertices by $u_1, v_1, u_2, v_2 \ldots v_l$. This regular spherical polygon naturally defines a two-dimensional polygon in $\mathbb{R}^3$ with the same vertices. Let n be a unit normal. Choose $v_{l+1} = u_{l+1} = -n$. Let μ, λ be equal-weight discrete measures from these vectors. First of all, the measure μ is not concentrated on a closed hemisphere. It is also not hard to convince oneself that, by choosing the polygon to be small enough, one can ensure that there are exactly two maximizers of the assignment functional f and g, where f is defined by:

$$f(j) = j \tag{8.9}$$

and g is:

$$g(j) = \begin{cases} j-1 & \text{if } 1 < j \leq l \\ l & \text{if } j = 1 \\ l+1 & \text{if } j = l+1. \end{cases} \tag{8.10}$$

It is also not hard to see that sets $\{u_1 \ldots u_l\}$ and $\{v_1 \ldots v_m\}$ have an edge normal loop which is given by piciewise linear path connecting points $x_i = v_i$.

Consider a small variation of vector u_2. As long as $\frac{A(f)}{A(g)} = 1$, the functional is not uniquely maximized. Holding all other vectors fixed, we can vary u_2 as a unit vector or as a point on a sphere so that $\frac{u_2 v_2}{u_2 v_1}$ is constant. This variation preserves $\frac{A(f)}{A(g)}$. Now suppose $c := \frac{u_2 v_2}{u_2 v_1}$, then for small perturbations:

$$\frac{u_2v_2}{u_2v_1} = c \Leftrightarrow \tag{8.11}$$
$$u_2(v_2 - v_1c) = 0.$$

So, in particular we can move u_2 along some geodesic so that $\frac{A(f)}{A(g)} = 1$. After some variation in u_2, we can pick v_2 and vary it similarly along geodesic preserving

$$\frac{u_2v_2}{u_3v_2} = c_2 \Leftrightarrow \tag{8.12}$$
$$v_2(u_2 - u_3c_2) = 0.$$

Note that we can only ensure these variations locally, as we would like to preserve the weak Aleksandrov relation, concentration condition, and since we want all other assignment functionals to remain less than $A(f) = A(g)$. All of these conditions are guaranteed by the openness and continuity of the respective sets and functions.

This illustration shows the geodesic variation of points on sphere so that solution to every variation doesn't exist. This variation preserves the edge-normal loop condition as well.

9. Appendix

These are results from our second paper on this subject, titled "The Gauss Image Problem with Weak Aleksanrov Condition," which are used in the proof of Theorem 5.1. We group them in this Appendix for the convenience.

Definition 9.1. Given two Borel measures μ and λ on S^{n-1}, a measure μ is weak Aleksandrov related to λ if $\mu(S^{n-1}) = \lambda(S^{n-1})$ and for each closed set $\omega \subset S^{n-1}$ contained in closed hemisphere, there exists an $\alpha \in (0, 1)$ such that

$$\mu(\omega) \leq \lambda(\omega_{\frac{\pi}{2}-\alpha}). \tag{9.1}$$

This definition for general measures agrees with the discrete weak Aleksandrov definition given earlier for discrete measures, by Proposition 3.6. Equipped with this definition, we show the following theorem, in "The Gauss Image Problem with Weak Aleksandrov Condition:"

Theorem 9.2. *[Theorem 1.4] Suppose μ is a discrete Borel measure not concentrated on a closed hemisphere, and λ is an absolutely continuous Borel measure. Suppose μ is weak Aleksandrov related to λ. Then, there exists a $K \in \mathcal{K}_o^n$ such that $\mu = \lambda(K, \cdot)$.*

Lemma 9.3. *[Lemma 4.6] Suppose μ and λ are given as in Theorem 9.2. Let α be a uniform constant for the weak Aleksandrov assumption. Then, there exists a polytope solution P to the Gauss Image problem of the form:*

$$P = \left(\bigcap_{i=1}^{m} H^-(\alpha_i, v_i)\right)^*, \tag{9.2}$$

such that $\frac{r_P}{R_p}$ is bounded from below by a constant $C_{\mu,\lambda}$, depending only on vectors v_i and the uniform weak Aleksandrov constant α. Besides being dependent on α, this constant is independent of λ.

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