THE GAUSS IMAGE PROBLEM WITH WEAK ALEKSANDROV CONDITION

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Abstract. We introduce a relaxation to the Aleksandrov relation assumption for the Gauss Image Problem. This new assumption turns out to be a necessary condition for two measures to be related by a convex body. We provide several properties of the new condition. A solution to the Gauss Image Problem is obtained for the case when one of the measures is assumed to be discrete and another measure is assumed to be absolutely continuous, under the new relaxed assumption.

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1. Introduction

The Gauss Image Problem, introduced in [13], is a natural extension to the classical Aleksandrov question of finding a body with the prescribed Aleksandrov's integral curvature [1–3]. This problem is a part of the study of Minkowski problems, a vital area of research in convex geometry. The study of these problems has led to the formulation of the log-Brunn-Minkowski inequality conjecture [5,12,17,21,36] and to the sharp affine L^p Sobolev inequality [28]. The latter has also inspired many other sharp affine isoperimetric inequalities [16,25,28]. Readers are referred to Chapters 8 and 9 of Schneider's textbook [37] for an introduction to Minkowski problems and to the articles [7, 10, 11, 18–20, 22, 24–27, 29, 32–34, 38, 40–44] for an overview of the recent developments. Additionally, we acknowledge the works [4,6,31,35,39] related to the regularity of Minkowski problems.

Given two measures μ and λ on S^{n-1}, the Gauss Image Problem asks about the existence of a convex body K, containing the origin in its interior, such that $\mu = \lambda(K, \cdot)$, where by $\lambda(K, \cdot)$ we denote the pullback of λ under the radial Gauss Image map of K: a composition of the multivalued Gauss map of K and the radial map of K. In fact, many significant measures can be described as pullbacks of a certain λ under the Gauss Image map. For instance, when λ is the spherical Lebesgue measure, $\lambda(K, \cdot)$ is known as Aleksandrov's integral curvature of the body K [3]. When λ is Federer's $(n-1)^{\text{th}}$ curvature measure, $\lambda(K, \cdot)$ is the surface area measure of Aleksandrov-Fenchel-Jessen [2]. Finally, the more recently defined the dual curvature measure is also a pullback of a certain λ under the Gauss Image map [19]. All of these examples motivate the necessity for a systematic study of how measures transfer to each other through the radial Gauss Image Map, that is, the Gauss Image Problem:

The Gauss Image Problem (Defined in [13]) *Suppose λ is a measure defined on the Lebesgue measurable subsets of S^{n-1}, and μ is a Borel measure on S^{n-1}. What are the necessary and sufficient conditions on λ and μ, so that there exists a convex body K with the origin in its interior such that*

$$\mu = \lambda(K, \cdot)? \tag{1.1}$$

If such a convex body exists, to what extent is it unique?

When λ is a spherical Lebesgue measure, we recover the original Aleksandrov problem, which Aleksandrov first studied in [1–3]. Different proofs of the Aleksandrov problem were given by Oliker [34] and Bertrand [8]. The L_p analogs of the Aleksandrov problem were considered by Huang, Lutwak, Yang, and Zhang in [18], by Mui in [30], and by Zhao in [42].

When one of the measures is assumed to be absolutely continuous, the Gauss Image Problem was studied in [13] by Böröczky, Lutwak, Yang, Zhang, and Zhao. There, the Aleksandrov relation was introduced to attack the problem:

Definition 1.1. Two Borel measures μ and λ on S^{n-1} are called Aleksandrov related if

$$\lambda(S^{n-1}) = \mu(S^{n-1}) > \mu(\omega) + \lambda(\omega^*) \tag{1.2}$$

for each compact, spherically convex set $\omega \subset S^{n-1}$, where the set ω^* is defined as a polar set:

$$\omega^* := \bigcap_{u\in\omega} \{v \in S^{n-1} : uv \leq 0\} \tag{1.3}$$

Equivalently, one can define two Borel measures μ and λ on S^{n-1} to be Aleksandrov related if $\mu(S^{n-1}) = \lambda(S^{n-1})$ and for each compact, spherically convex set $\omega \subset S^{n-1}$,

$$\mu(\omega) < \lambda(\omega_{\frac{\pi}{2}}), \tag{1.4}$$

where

$$\omega_{\frac{\pi}{2}} := \bigcup_{u\in\omega} \{v \in S^{n-1} : u \cdot v > 0\}. \tag{1.5}$$

With this new condition, the following solution to the Gauss Image Problem was obtained [13]:

Theorem 1.2 (K. J. Böröczky, E. Lutwak, D. Yang, G. Y. Zhang and Y. M. Zhao [13])**.** *Suppose μ and λ are Borel measures on S^{n-1}, and λ is absolutely continuous. If μ and λ are Aleksandrov related, then there exists a body $K \in \mathcal{K}^n_o$, such that $\mu = \lambda(K, \cdot)$.*

Moreover, if one of the measures is assumed to be absolutely continuous and strictly positive on open sets, it was shown that the Aleksandrov relation is a necessary assumption for the existence of a solution to the Gauss Image Problem. In this case, a solution to the Gauss Image Problem was shown to be unique up to a dilation. We refer the reader to [13] for this result and an introduction to the Gauss Image Problem. Additionally, let us also mention Theorem 1.7 and Remark 4.9 in Bertrand [8], which also imply Theorem 1.2 using a very different method.

While the Aleksandrov relation is a natural assumption when one of the measures is positive on open sets, it turns out that there are numerous examples of measures μ and λ satisfying $\mu = \lambda(K, \cdot)$ that are not Aleksandrov related. For instance, consider $K = B^n$ and μ and λ to be any even, absolutely continuous, equal measures supported on small symmetric spherical caps ω, where $\omega \subset S^{n-1}$ is a cap around the north pole and $-\omega \subset S^{n-1}$ is a cap around the south pole. Then, $\mu = \lambda = \lambda(K, \cdot)$, and $\mu(\omega) + \lambda(\omega^*) = \lambda(S^{n-1})$, which violates the Aleksandrov relation. Moreover, starting with the body $K = B^n$, we can perturb it along the equator while preserving the convexity. We thereby obtain a family of convex bodies such that every member still solves the Gauss Image Problem. This observation indicates that, in general, the solution to the Gauss Image Problem may be highly non-unique.

Based on these considerations, we introduce a relaxation of the Aleksandrov relation for the Gauss Image Problem. This relaxation turns out to be a necessary assumption for the two measures to be related by a convex body. That is, for the existence of a convex body K with origin in its interior such that $\mu = \lambda(K, \cdot)$. See Proposition 3.1.

Definition 1.3. Given Borel measures μ and λ on S^{n-1}, we say that μ is weakly Aleksandrov related to λ if $\mu(S^{n-1}) = \lambda(S^{n-1})$ and for each closed set $\omega \subset S^{n-1}$, contained in a closed hemisphere, there exists $\alpha \in (0, \frac{\pi}{2})$ such that

$$\mu(\omega) \le \lambda(\omega_{\frac{\pi}{2}-\alpha}), \tag{1.6}$$

where

$$\omega_{\frac{\pi}{2}-\alpha} := \bigcup_{u\in\omega} \{v \in S^{n-1} : u \cdot v > \cos(\frac{\pi}{2} - \alpha)\}. \tag{1.7}$$

Besides showing that the weak Aleksandrov relation is a necessary assumption for the existence of a solution to the Gauss Image Problem, we also show that the classical Aleksandrov relation implies the weak Aleksandrov relation. The nature of the constant α is addressed in

Section 3. In particular, if $\mu = \lambda(K, \cdot)$, then constant α in the weak Aleksandrov condition is closely related to the inner to outer radius ratio of the body K. See Proposition 3.1 and the discussion after it.

Now, with an appropriate necessary condition, we are ready to state the main result of the paper. In the following, a measure μ is a discrete measure if it can be expressed as

$$\mu = \sum_{i=1}^{m} \mu_i \delta_{v_i} \tag{1.8}$$

where μ_i are some positive coefficients, and δ_{v_i} is a Dirac measure of the set $\{v_i\}$. For the measure μ, we also define the set of polytopes P_μ as

$$\mathcal{P}_\mu = \operatorname{conv}\{\beta_i v_i \mid 1 \leq i \leq m\} \tag{1.9}$$

where $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{R}^m_{>0}$.

Theorem 1.4. *Let μ be a discrete Borel measure that is not concentrated on a closed hemisphere, and let λ be an absolutely continuous Borel measure. If μ is weakly Aleksandrov related to λ, then there exists a polytope $P \in P_\mu$ such that $\mu = \lambda(K, \cdot)$.*

In particular, we establish that given a discrete measure μ and an absolutely continuous measure λ, the weak Aleksandrov relation is a necessary and sufficient condition for the existence of a solution to the Gauss Image Problem.

In conclusion, we would like to comment on the differences between the methods introduced in this paper and those presented in [13]. The proof of Theorem 1.2 in [13] has the following structure: First, it is shown that any convex body that maximizes the specific functional on convex bodies (defined below, see (2.15)) is a solution to the Gauss Image Problem. Then, by analyzing the classical Aleksandrov relation for specific measures, it is proven that any sequence of convex bodies maximizing this functional exhibits a bound on the inner to outer radius ratio of its elements. This bound is arguably the most challenging aspect of the paper [13]. From the Blaschke selection theorem, we deduce that this sequence contains a convergent subsequence that converges to a non-degenerate convex body K maximizing the functional (2.15). The limiting body K, in turn, solves the Gauss Image Problem.

The main challenge and difference in the proof of Theorem 1.4, as compared to the main result of [13], is that the weak Aleksandrov relation does not impose a bound on the inner to outer radius ratio for the possible solution, unlike its strong counterpart. Going back to the previously mentioned example of spherical caps, for any scalars $\lambda_1, \lambda_2 > 0$, define K_{λ_1,λ_2} to be the convex hull in $\mathbb{R}^n$ of $\lambda_1\omega \subset \lambda_1 S^{n-1}$ and $-\lambda_2\omega \subset \lambda_2 S^{n-1}$. Note that any K_{λ_1,λ_2} is a solution to $\mu = \lambda(K, \cdot)$, where μ and λ are defined as before. (This is true because the normal cones of K_{λ_1,λ_2} do not change for radial directions contained in the support of μ, when we vary λ_1 and λ_2. See Preliminaries for the definitions.) Hence, in contrast to the classical Aleksandrov relation assumption, the solution body may contain parts that one can dilate independently. Consequently, unlike the case when the classical Aleksandrov relation is assumed, a sequence of convex bodies, say $(K_\mathfrak{n})_{\mathfrak{n}=1}^\infty$ such that $K_\mathfrak{n} \subset rB^n$ for all $\mathfrak{n}$ and some r, may maximize the functional while converging to a degenerate convex body. This makes the proof of the main theorem in this paper vastly differ from that in [13], as not every sequence of normalized convex bodies maximizing the functional is suitable for the proof. To construct this sequence and overcome these challenges, we invoke a process that we call the partial rescaling of convex bodies. See (4.17).

It would be very interesting to see whether one could prove the result of the Theorem 1.2, the main results of [13], under the weak Aleksandrov condition instead of the classical Aleksandrov condition. Our paper can be viewed as a step towards this direction. We state this in the Conjecture 6.3.

2. Preliminaries

By $\mathcal{K}^n$ we denote the set of convex bodies (compact, convex subsets with nonempty interior in $\mathbb{R}^n$). By $\mathcal{K}^n_o \subset \mathcal{K}^n$ we denote those convex bodies that contain the origin in their interiors. Given $K \in \mathcal{K}^n_o$, let $x \in \partial K$ be a boundary point. *The normal cone* at x is defined by

$$N(K,x) = \{v \in S^{n-1} : (y-x)\cdot v \leq 0 \text{ for all } y \in K\}, \tag{2.1}$$

which parametrizes all normals at a given boundary point. For $K \in \mathcal{K}^n_o$, *the radial map* $r_K : S^{n-1} \to \partial K$ of K is defined for $u \in S^{n-1}$ by $r_K(u) = ru \in \partial K$, where $r > 0$. Given a subset ω of S^{n-1}, the *radial Gauss Image* of ω is defined as follows:

$$\boldsymbol{\alpha}_K(\omega) = \bigcup_{x \in r_K(\omega)} N(K,x) \subset S^{n-1}. \tag{2.2}$$

The radial Gauss Image map, $\boldsymbol{\alpha}_K$, maps sets of S^{n-1} to sets of S^{n-1}. Outside of a spherical Lebesgue measure zero set, the multivalued map $\boldsymbol{\alpha}_K$ is singular valued. It is known that $\boldsymbol{\alpha}_K$ maps Borel measurable sets to Lebesgue measurable sets. See [37] for both of these results. We denote the restriction of $\boldsymbol{\alpha}_K$ to the corresponding singular valued map by α_K. For additional details, we refer the reader to [13].

The radial function $\rho_K : S^{n-1} \to \mathbb{R}$ is defined by:

$$\rho_K(u) = \max\{a : au \in K\}. \tag{2.3}$$

In this case, $r_K(u) = \rho_K(u)u$. *The support function* is defined by:

$$h_K(x) = \max\{x \cdot y : y \in K\}. \tag{2.4}$$

For $K \in \mathcal{K}^n_o$, we define its *polar body* K^* by $h_{K^*} := \frac{1}{\rho_K}$.

We denote by r_K the radius of the largest ball contained in K and centered at o. Similarly, we denote R_K to be the radius of the smallest ball containing K and centered at o. We will refer to r_K as *the inner radius* of body K, to R_K as *the outer radius* of K, and to the ratio $\frac{r_K}{R_k}$ as *the inner to outer radius ratio* of K.

It is important to note that for any $K \in \mathcal{K}^n_o$, the following identity holds:

$$\min \rho_K = \min h_K = r_K \leq R_k = \max \rho_K = \max h_K. \tag{2.5}$$

The support hyperplane to K with an outer unit normal $v \in S^{n-1}$ is defined as

$$H_K(v) = \{x : x \cdot v = h_K(v)\}. \tag{2.6}$$

Let $H^-(\alpha, v) = \{x : x \cdot v \leq \alpha\}$ and $H(\alpha, v) = \{x : x \cdot v = \alpha\}$. Given a set $S \subset \mathbb{R}^n$ we write its convex hull as

$$\operatorname{conv}(S). \tag{2.7}$$

For a set $\omega \subset S^{n-1}$, we define $\operatorname{cone}\omega$ as the *cone that* ω *generates*, that is

$$\operatorname{cone}\omega = \{tu : t \geq 0 \text{ and } u \in \omega\}. \tag{2.8}$$

We say that $\omega \subset S^{n-1}$ is *spherically convex* if the cone that ω generates is a nonempty, proper, convex subset of $\mathbb{R}^n$. Therefore, a spherically convex set in S^{n-1} is always nonempty and contained in a closed hemisphere of S^{n-1}. Given $\omega \subset S^{n-1}$ contained in a closed hemisphere, *the polar set* ω^* is defined by:

$$\omega^* = \bigcap_{u \in \omega} \{v \in S^{n-1} : u \cdot v \leq 0\}. \tag{2.9}$$

We note that the polar set is always spherically convex. If $\omega \subset S^{n-1}$ is a closed set, we define its *outer parallel set* ω_α for some $\alpha \in (0, \frac{\pi}{2}]$ to be

$$\omega_\alpha = \bigcup_{u \in \omega} \{v \in S^{n-1} : u \cdot v > \cos \alpha\}. \tag{2.10}$$

As mentioned previously, $\boldsymbol{\alpha}_K$ maps Borel measurable sets to Lebesgue measurable sets. Given a Borel measure λ we, as in [13], define *the Gauss Image measure of* λ *via* K as

$$\lambda(K, \omega) := \lambda(\boldsymbol{\alpha}_K(\omega)) \tag{2.11}$$

for each Borel $\omega \in S^{n-1}$. Note, however, that the naming is a bit misleading as, in general, $\lambda(K, \cdot)$ does not necessarily have to be a measure. For example, takes K to be a square centered at the origin with sides perpendicular to vectors u_1, u_2, u_3, u_3. Let $\lambda = \sum_{i=1}^{4} \delta_{u_i}$ where δ_{u_i} are Dirac measures of sets $\{u_i\}$. Let vectors v_1 and v_2 be such that $r_K(v_1), r_K(v_2)$ are in the interior of the side of K perpendicular to u_1. Then,

$$\boldsymbol{\alpha}_K(\{v_1\}) = \boldsymbol{\alpha}_K(\{v_2\}) = \boldsymbol{\alpha}_K(\{v_1, v_2\}) = \{u_1\}. \tag{2.12}$$

Implying that:

$$1 = \lambda(K, \{v_1\}) = \lambda(K, \{v_2\}) = \lambda(K, \{v_1, v_2\}), \tag{2.13}$$

which establishes that $\lambda(K, \cdot)$ is not countably additive.

On the other hand, if λ is absolutely continuous, which is the case of this work, $\lambda(K, \cdot)$ is always a measure. For this and related results, see [13]. We also point out Lemma 3.3 in [13], which states that:

Lemma 2.1. *If λ is an absolutely continuous Borel measure, and $K \in \mathcal{K}_o^n$, then*

$$\int_{S^{n-1}} f(u) d\lambda(K, \cdot) = \int_{S^{n-1}} f(\alpha_K(v)) d\lambda(v) \tag{2.14}$$

for each bounded Borel measurable function $f : S^{n-1} \to \mathbb{R}$.

We note that if for a given μ and λ, there exists $K \in \mathcal{K}_o^n$ such that $\mu = \lambda(K, \cdot)$, then we say that measures μ and λ are *related by the convex body* K. For $K \in \mathcal{K}_o^n$ and λ absolutely continuous, we define the functional $\Phi(K, \mu, \lambda)$ by

$$\Phi(K, \mu, \lambda) := \int \log \rho_K d\mu + \int \log \rho_{K^*} d\lambda. \tag{2.15}$$

Sometimes, we will write $\Phi(K)$, suppressing some notation. Note that $\Phi(K, \mu, \lambda) = \Phi_{\mu,\lambda}(K^*)$ in the notation of [13]. This functional is intimately associated with the Gauss Image Problem. For example, Theorem 8.2 in [13] shows that if μ is a Borel measure and λ is an absolutely continuous Borel measure such that

$$\Phi(K, \mu, \lambda) = \sup_{K' \in \mathcal{K}_o^n} (K', \mu, \lambda) \tag{2.16}$$

for $K \in \mathcal{K}_o^n$, then $\mu = \lambda(K, \cdot)$. It is important to stress that:

$$\begin{aligned} &\text{If } \mu = \lambda(K, \cdot), \text{ then } \mu = \lambda(cK, \cdot) \text{ for any } c > 0. \\ &\Phi(K, \mu, \lambda) = \Phi(cK, \mu, \lambda) \text{ for any } c > 0. \end{aligned} \tag{2.17}$$

That is, the nature of the problem is not sensitive to the rescaling of the convex bodies.

The Aleksandrov relation, as well as the weak Aleksandrov relation, were defined in the Introduction. We simply note the interchangeable use of the terms "Aleksandrov condition" and "Aleksandrov relation".

A measure μ is called discrete if it takes the form:

$$\mu = \sum_{i=1}^{m} \mu_i \delta_{v_i} \tag{2.18}$$

where δ_{v_i} are Dirac measures of sets $\{v_i\}$ containing a single vector $v_i \in S^{n-1}$ and μ_i are strictly positive coefficients. Aside from Proposition 3.3, the measure μ will always be assumed to be discrete and written as in (2.18) with letters v and m reserved specifically for μ.

Given a discrete measure μ not concentrated on a closed hemisphere, we define $\mathcal{P}_\mu$ to be the set of the convex hull of points $\{\beta_i v_i\}$ with $\beta_i > 0$. In other words,

$$\mathcal{P}_\mu = \operatorname{conv}\{\beta_i v_i \mid 1 \leq i \leq m\} \tag{2.19}$$

with $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m_{>0}$. Given any $P \in \mathcal{P}_\mu$, since μ is not concentrated on a closed hemisphere, P contains the origin in its interior. Therefore, $\mathcal{P}_\mu \subset \mathcal{K}_o^n$. Moreover, any $P \in \mathcal{P}_\mu$ is a polytope, such that each vertex of P is located in a radial direction v_i for some $i \in \{1, \ldots, m\}$. Note, however, that sometimes a polytope $P \in \mathcal{P}_\mu$ might have fewer than m vertices corresponding to some $\beta_j v_j$ contained inside the convex hull of the remaining points.

The next part of notations can be viewed as a discrete analog to the standard concepts of the support function and the Wulff shape. Given $P \in \mathcal{P}_\mu$, we define its representation to be an m-tuple of positive numbers

$$\alpha = (\alpha_1, \ldots, \alpha_m) := (h_{P^*}(v_1), \ldots, h_{P^*}(v_m)). \tag{2.20}$$

Note that if α is the representation of P then

$$\begin{aligned} P &= \operatorname{conv}\{\frac{v_i}{\alpha_i} \mid 1 \leq i \leq m\} \\ P^* &= \bigcap_{i=1}^{m} H^-(\alpha_i, v_i) \end{aligned} \tag{2.21}$$

Conversely, suppose we start with some $m-$tuple of positive numbers, γ. We define $P_\gamma \in \mathcal{P}_\mu$ to be the following:

$$P_\gamma = (\bigcap_{i=1}^{m} H^-(\gamma_i, v_i))^*. \tag{2.22}$$

We call such P a *dual Wulff Shape* of the $m-$tuple γ. We refer to the representation of P_γ as the *Wulff tuple* of γ. In particular, one has that if α is a Wulff tuple of γ, then

$$\alpha_i \leq \gamma_i. \tag{2.23}$$

Moreover, if $\alpha_i < \gamma_i$, then the facet of P^*_γ in the direction v_i is degenerate.

If a polytope has an index a, such as P_a, we are going to write its coefficients in representation as:

$$\alpha_a = (\alpha_{a,1}, \dots, \alpha_{a,m}) \tag{2.24}$$

Given $P \in \mathcal{P}_\mu$ with its representation denoted by the m-tuple α and a nonempty and not full indexing set $I \subset \{1 \dots m\}$, we will denote by U, L, U^*, L^* the following quantities:

$$\begin{aligned} U &:= \max_{i\in I} \alpha_i, \\ L &:= \min_{i\in I} \alpha_i, \\ U^* &:= \max_{i\notin I} \alpha_i, \\ L^* &:= \min_{i\notin I} \alpha_i. \end{aligned} \tag{2.25}$$

It will usually be the case that:

$$0 < L^* \leq U^* \leq L \leq U = 1. \tag{2.26}$$

If a polytope has an index t, such as P_t, we are going to write

$$P_t, \alpha_t, L_t, U_t, L_t^*, U_t^*. \tag{2.27}$$

Finally, in the proofs of Lemma 6.1 and Lemma 6.2, we will need to use different index sets. In the context of these Lemmas, we will denote the same quantities for a specific index set I as:

$$L_a(I), U_a(I), L_a^*(I), U_a^*(I). \tag{2.28}$$

We use the books of Schneider [37] as our standard reference. The books of Gruber and Gardner are also good alternatives [14, 15].

3. Weak Aleksandrov Condition

Let us start by showing that the weak Aleksandrov relation is a necessary condition for Borel measures to be related by a convex body.

Proposition 3.1. *Given $K \in \mathcal{K}_o^n$, suppose λ and $\lambda(K,\cdot)$ are Borel measures. Then, λ is weakly Aleksandrov related to $\lambda(K,\cdot)$.*

Remark. Note that if λ is an absolutely continuous Borel measure, then $\lambda(K,\cdot)$ automatically becomes a Borel measure. For more details, see Preliminaries section and Lemma 3.3 in [13]. △

Proof. Since $K \in \mathcal{K}_o^n$, there exists $c > 0$ such that $\frac{r_K}{R_K} > c$. Consider some $u \in S^{n-1}$ and $v \in \boldsymbol{\alpha}_K(u)$. Then,

$$r_K \leq h_K(v) = \rho_K(u)uv \leq R_K uv \tag{3.1}$$

Hence, $c < \frac{r_K}{R_K} \leq uv$. Therefore, for each $u \in S^{n-1}$, we have:

$$\boldsymbol{\alpha}_K(u) \subset u_{\arccos(c)} \subset u_{\frac{\pi}{2}-\alpha}, \tag{3.2}$$

for some α, where $0 < \alpha < \frac{\pi}{2}$. Therefore, for any closed set ω contained in a closed hemisphere, since $\omega_{\frac{\pi}{2}-\alpha} = \bigcup_{u\in\omega} u_{\frac{\pi}{2}-\alpha}$, we obtain:

$$\lambda(K,\omega) = \lambda(\boldsymbol{\alpha}_K(\omega)) \leq \lambda(\omega_{\frac{\pi}{2}-\alpha}). \tag{3.3}$$

□

In particular, the above proof shows that given measures λ and $\lambda(K, \cdot)$, the constant α in the weak Aleksandrov relation does not depend on the choice of a closed set ω contained in a closed hemisphere. Moreover, the constant α in the above proof encompasses the bound on $\frac{r_K}{R_K}$. That is, the bound on the inner to outer radius ratio for body K. The following Proposition is the step in the opposite direction. Recall the notation for a discrete measure μ in (2.18).

Proposition 3.2. *Suppose that a discrete Borel measure μ is weakly Aleksandrov related to a Borel measure λ. Then there exists a uniform constant $\alpha \in (0, \frac{\pi}{2})$ such that for any $\omega \subset S^{n-1}$, a closed set contained in a closed hemisphere:*

$$\mu(\omega) \leq \lambda(\omega_{\frac{\pi}{2}-\alpha}) \tag{3.4}$$

Remark. We refer to this α as *the uniform weak Aleksandrov constant* for measures μ and λ. △

Proof. Consider all possible $I \subset \{1 \dots m\}$ such that $\{v_i\}_{i \in I}$ are contained in a closed hemisphere. Let $\omega^I = \bigcup_{i \in I} v_i$. Since μ and λ are weak Aleksandrov related, we have $\mu(\omega^I) \leq \lambda(\omega^I_{\frac{\pi}{2}-\alpha_I})$ for some α_I. Since there are only finitely many of those I satisfying the assumption, we can choose $\alpha > 0$ to be the minimum of the α_I's. Now for any closed set ω contained in a closed hemisphere, we obtain that for some $I \subset \{1 \dots m\}$:

$$\mu(\omega) = \mu(\omega \cap \{v_i\}_{i \in \{1 \dots m\}}) = \mu(\omega^I) \leq \lambda(\omega^I_{\frac{\pi}{2}-\alpha}) \leq \lambda(\omega_{\frac{\pi}{2}-\alpha}), \tag{3.5}$$

where the last step follows from set inclusion. □

Finally, we note that the classical Aleksandrov relation easily implies the weak Aleksandrov relation. In the following, μ is not necessarily a discrete measure.

Proposition 3.3. *Suppose that a Borel measure μ is Aleksandrov related to Borel measure λ. Then, μ is weakly Aleksandrov related to λ.*

Proof. Since μ is Aleksandrov related to λ, for each compact, spherically convex set $\omega \subset S^{n-1}$, we obtain:

$$\mu(\omega) < \lambda(S^{n-1}) - \lambda(\omega^*) = \lambda(\omega_{\frac{\pi}{2}}). \tag{3.6}$$

Now, consider any closed set γ contained in a closed hemisphere. Let

$$\omega = \langle \gamma \rangle := S^{n-1} \cap \operatorname{conv}(\operatorname{cone} \gamma), \tag{3.7}$$

where $\operatorname{conv}(\operatorname{cone} \gamma)$ denotes the convex hull of $\operatorname{cone} \gamma$. Note that the convex hull of any set $S \in \mathbb{R}^n$ is given by all finite convex combinations of elements in S. Thus, recalling the definition of a cone, we obtain the following: for each $v \in \omega$, there exist vectors $\{v_i\}_{i \in I} \subset \gamma$ with I finite such that

$$v = \sum_{i \in I} \sigma_i v_i \text{ where } \sigma_i > 0. \tag{3.8}$$

Choose any $u \in \omega_{\frac{\pi}{2}}$. Then for some $v \in \omega$, $uv > 0$. Hence, for some $\{v_i\}_{i \in I} \subset \gamma$ with I finite,

$$uv = \sum_{i \in I} \sigma_i u v_i > 0 \text{ where } \sigma_i > 0. \tag{3.9}$$

Thus, at least for one $i \in I$, we have that $uv_i > 0$. Hence, $u \in v_{i\frac{\pi}{2}} \subset \gamma_{\frac{\pi}{2}}$. We have that $\omega_{\frac{\pi}{2}} \subset \gamma_{\frac{\pi}{2}}$. Thus, we obtain the following chain of inequalities:

$$\mu(\gamma) \leq \mu(\omega) < \lambda(\omega_{\frac{\pi}{2}}) \leq \lambda(\gamma_{\frac{\pi}{2}}). \tag{3.10}$$

By the continuity of measures, $\lambda(\gamma_{\frac{\pi}{2}-\alpha}) \to \lambda(\gamma_{\frac{\pi}{2}})$ as $\alpha \to 0$. Hence, for a given closed set γ contained in a closed hemisphere, there exists an α such that

$$\mu(\gamma) < \lambda(\gamma_{\frac{\pi}{2}-\alpha}). \tag{3.11}$$

The weak Aleksandrov condition follows. □

One might wonder whether we can define the weak Aleksandrov relation merely by restricting the definition to a collection of compact spherically convex sets instead of closed sets contained in a closed hemisphere. We have not investigated this question, leaving it to the reader if they are interested.

4. Essential Estimates and The Partial Rescaling of a Polytope

For the rest of the paper, we will assume that μ is a discrete Borel measure written as in (2.18), which is not concentrated on a closed hemisphere. We will also assume that λ is an absolutely continuous measure. We begin with the following lemma, which enables us to concentrate our attention on polytopes only.

Lemma 4.1. *Given any $K \in \mathcal{K}_o^n$, there exists a polytope $P \in \mathcal{P}_\mu$ such that:*

$$\Phi(P, \mu, \lambda) \geq \Phi(K, \mu, \lambda). \tag{4.1}$$

Proof. Choose any $K \in \mathcal{K}_o^n$. Define P as:

$$P := \operatorname{conv}\{\rho_K(v_i)v_i \mid 1 \leq i \leq m\}. \tag{4.2}$$

Clearly, $P \in \mathcal{P}_\mu$. Moreover, since P is a convex hull of a subset of K, $P \subset K$. Hence, $h_P \leq h_k$, which implies that:

$$\int \log \rho_{K^*} d\lambda \leq \int \log \rho_{P^*} d\lambda. \tag{4.3}$$

Simultaneously, by the definition of P, for any i we have that $\rho_P(v_i) \geq \rho_K(v_i)$. Since $P \subset K$, we also have that $\rho_P(v_i) \leq \rho_K(v_i)$. Therefore, for all i, $\rho_P(v_i) = \rho_K(v_i)$, and, thus,

$$\int \log \rho_K d\mu = \int \log \rho_P d\mu. \tag{4.4}$$

Combining both equations (4.2) and (4.4), we obtain that $\Phi(K, \mu, \lambda) \leq \Phi(P, \mu, \lambda)$. □

Theorem 8.2 in [13] shows that if $K \in \mathcal{K}_o^n$ maximizes the functional $\Phi(\cdot, \mu, \lambda)$ for a Borel measure μ and an absolutely continuous measure λ, then K solves the Gauss Image Problem. Thus, to prove Theorem 1.4, it is sufficient to establish the existence of a $K \in \mathcal{K}_o^n$ that maximizes the functional. Lemma 4.1 permits us to restrict bodies to polytopes of the above form, allowing us to work exclusively within the class P_μ.

We start with some Lemmas concerning the class P_μ. For the rest of the article, we are going to work with the notation defined in (2.19)-(2.25). Recall that for a given $P \in \mathcal{P}_\mu$, we call an m-tuple α to be a representation of P if it is equal to

$$\alpha = (\alpha_1, \dots, \alpha_m) = (h_{P^*}(v_1), \dots, h_{P^*}(v_m)). \tag{4.5}$$

Lemma 4.2. *Given $P \in \mathcal{P}_\mu$, let α be its representation. Then, for any $u \in S^{n-1}$:*

$$\rho_{P^*}(u) = \min_{\forall i \ s.t. \ uv_i > 0} \frac{\alpha_i}{uv_i}. \tag{4.6}$$

Moreover,

$$\rho_{P^*}(u) = \frac{\alpha_i}{uv_i} \tag{4.7}$$

if and only if $r_{P^*}(u) \in H_{P^*}(v_i)$.

Proof. Recall from (2.21) that P^* can be written as,

$$P^* = \bigcap_{i=1}^{m} H^-(\alpha_i, v_i). \tag{4.8}$$

Fix some $u \in S^{n-1}$. Suppose for some i, $uv_i \leq 0$. Then $H^-(\alpha_i, v_i)$ contains the entire ray in the direction of u starting at the center. Suppose now for a given i, $uv_i > 0$. Then a ray in the direction of u starting at the center intersects the hyperplane $H(\alpha_i, v_i)$. In this case, it is straightforward to verify that the distance from the center to the intersection will be:

$$\frac{\alpha_i}{uv_i} \tag{4.9}$$

Therefore, looking back at the equation (4.8), we obtain that

$$\rho_{P^*}(u) = \min_{\forall i \ \text{s.t.} uv_i > 0} \frac{\alpha_i}{uv_i}. \tag{4.10}$$

The last part of the statement follows from equation (4.10) and (4.8). ☐

The following lemma is a core estimate, which will later be used to properly rescale the polytopes without decreasing the value of the functional.

Lemma 4.3. *Given $P \in \mathcal{P}_\mu$, let α be its representation. If we are given a nonempty and not full indexing set $I \subset \{1 \dots m\}$, with $U^* < L$, then:*

$$\bigcup_{i \notin I} (v_i)_{\arccos \frac{U^*}{L}} \subset \boldsymbol{\alpha}_P\Big(\bigcup_{i \notin I} v_i\Big). \tag{4.11}$$

Proof. Suppose that for a given direction $u \in S^{n-1}$, we have $\rho_{P*}(u) < L$. Then, using Lemma 4.2 we obtain

$$\min_{\forall i \ \text{s.t.} \ uv_i > 0} \frac{\alpha_i}{uv_i} = \rho_{P^*}(u) < L. \tag{4.12}$$

Suppose the minimum is achieved for some index j. Then, from the previous equation, using the definition of L, we obtain:

$$\frac{\alpha_j}{uv_j} < L = \min_{i \in I} \alpha_i. \tag{4.13}$$

Thus, since $0 < uv_j \leq 1$, we have that $j \notin I$. Now, applying the second part of Lemma 4.2 we obtain that

$$\begin{aligned} r_{P^*}(u) \in H_{P^*}(v_j) &\Leftrightarrow \\ r_P(v_j) \in H_P(u) &\Leftrightarrow \\ u \in \boldsymbol{\alpha}_P(v_j)&. \end{aligned} \tag{4.14}$$

Since $j \notin I$, we obtain $u \in \boldsymbol{\alpha}_P(\bigcup_{i \notin I} v_i)$.

Thus, we have established that if for a given direction $u \in S^{n-1}$, we have $\rho_{P*}(u) < L$, then $u \in \boldsymbol{\alpha}_P(\bigcup_{i \notin I} v_i)$. Now, pick any $u \in (v_j)_{\arccos \frac{U^*}{L}}$ for some $j \notin I$. We obtain $uv_j > \frac{U^*}{L}$. Thus, combining this with equation (4.12) and the fact that $j \notin I$, we obtain:

$$\rho_{P^*}(u) = \min_{\forall i \text{ s.t. } uv_i > 0} \frac{\alpha_i}{uv_i} \leq \frac{\alpha_j}{uv_j} \leq \frac{U^*}{uv_j} < L. \tag{4.15}$$

Thus, $\rho_{P*}(u) < L$ and the claim follows from the first part of the proof.

□

In the upcoming proof of Theorem 1.4, we will utilize what we refer to as *the partial rescaling of a polytope*. Let us now describe this construction. Suppose we are given a polytope $P \in \mathcal{P}_\mu$, along with a nonempty and not full indexing set $I \in \{1 \dots m\}$. Let α denote the representation of P. Recall that P and P^* can be written as

$$\begin{aligned} P &= \operatorname{conv}\{\frac{v_i}{\alpha_i} \mid 1 \leq i \leq m\} \\ P^* &= \bigcap_{i=1}^{m} H^-(\alpha_i, v_i) \end{aligned} \tag{4.16}$$

We would like to rescale the half spaces that correspond to the index set I in the second formula by a factor t. We call this a *partial rescaling of polytope P with index set I*. In terms of the preceding formula, this can be written as

$$\begin{aligned} P_t &= \operatorname{conv}\big(\{\frac{v_i}{\alpha_i} \mid i \in I\} \cup \{\frac{v_i}{t\alpha_i} \mid i \notin I\}\big) \\ P_t^* &= \bigcap_{i \in I} H^-(\alpha_i, v_i) \bigcap_{i \notin I} H^-(t\alpha_i, v_i). \end{aligned} \tag{4.17}$$

In fact, P_t is the dual Wulff Shape of the m-tuple γ_t defined by:

$$\begin{aligned} \gamma_{t,i} &= \alpha_i \text{ if } i \in I \\ \gamma_{t,i} &= t\alpha_i \text{ if } i \notin I \end{aligned} \tag{4.18}$$

See the Preliminaries section for the definitions. We will always assume that $t \in (0, 1]$. The most important point to make about the partial rescaling is that the representation of P_t is not necessarily equal to γ_t. For example, if the set $\{v_i \mid i \notin I\}$ is not contained in a closed hemisphere, P_t^* will approach center in the Hausdorff distance as $t \to 0$, and, thus, for all $i \in \{1 \dots m\}$ we have $\alpha_{t,i} \to 0$ while $\gamma_{t,i} = \alpha_i$ remains constant. If α_t is the representation of P_t, we can only claim that:

$$\alpha_{t,i} \leq \gamma_{t,i}. \tag{4.19}$$

The following lemma characterizes the behavior of the partial rescaling. It can be seen as a discrete analog to the classical results about Wulff Shapes.

Lemma 4.4. *Suppose $P \in \mathcal{P}_\mu$ and $I \subset \{1 \dots m\}$ is a nonempty and not full indexing set. Let α be its dual representation. Consider the $m-$tuple γ_t defined by:*

$$\begin{aligned} \gamma_{t,i} &= \alpha_i \textit{ if } i \in I \\ \gamma_{t,i} &= t\alpha_i \textit{ if } i \notin I. \end{aligned} \tag{4.20}$$

Let P_t be the dual Wulff Shape of γ. Let α_t denote its representation. Then,

$$\begin{aligned} t\alpha_i \le \alpha_{t,i} \le \alpha_i \ \textit{for } i \in I \\ \alpha_{t,i} = \gamma_i \ \textit{if } i \notin I. \end{aligned} \tag{4.21}$$

Proof. Let $i \notin I$. By the definition of the representation,

$$\alpha_{t,i} = h_{P^*_{\gamma_t}}(v_i). \tag{4.22}$$

From (4.17), we obtain that $tP^* \subset P^*_{\gamma_t}$. Thus,

$$\alpha_{t,i} = h_{P^*_{\gamma_t}}(v_i) \ge th_{tP^*}(v_i) = t\alpha_i. \tag{4.23}$$

On the other hand, from the definition of the dual Wulff Shape we have that

$$P^*_{\gamma_t} \subset P^* \cap H^-(t\alpha_i, v_i), \tag{4.24}$$

which implies that

$$\alpha_{t,i} = h_{P^*_{\gamma_t}}(v_i) \le t\alpha_i. \tag{4.25}$$

If $i \in I$, then, as $tP^* \subset P^*_{\gamma_t} \subset P^*$, we obtain

$$t\alpha_i \le \alpha_{t,i} \le \alpha_i. \tag{4.26}$$

□

5. The Partial Rescaling of a Polytope and the Functional

We now turn our attention to the key lemma related to the partial rescaling. Under the assumption of the weak Aleksandrov relation, as we have noted, we can no longer claim that for any sequence maximizing the functional, there is a lower bound on the inner to outer radius ratio. The following Lemma provides us with the tool to overcome this difficulty. It establishes that, with proper assumptions, the partial rescaling does not decrease the value of the functional.

Lemma 5.1. *Suppose $P \in \mathcal{P}_\mu$ and $I \subset \{1 \ldots m\}$ is a nonempty and not full indexing set. Let α be its dual representation. Consider the m-tuple γ_t defined by*

$$\begin{aligned} \gamma_{t,i} = \alpha_i \ \textit{if } i \in I \\ \gamma_{t,i} = t\alpha_i \ \textit{if } i \notin I \end{aligned} \tag{5.1}$$

Let P_t be the dual Wulff shape of γ_t. Given some $t_0 < 1$, suppose that

$$\mu(\bigcup_{i \notin I} v_i) \ge \lambda(\boldsymbol{\alpha}_{P_{t_0}}(\bigcup_{i \notin I} v_i)), \tag{5.2}$$

then $\Phi(P_{t_0}) \ge \Phi(P)$.

Proof. Before we start to compare $\Phi(P_{t_0})$ with $\Phi(P)$, let us first analyze the behavior of the radial functions under the partial rescaling. Let t be any value between t_0 and 1. First, we claim that for any $u \in S^{n-1}$:

$$\rho_{P^*_t}(u) = \min_{\forall i \text{ s.t.} uv_i > 0} \{\frac{\alpha_i}{uv_i} \text{ if } i \in I \text{ or } \frac{t\alpha_i}{uv_i} \text{ if } i \notin I\}. \tag{5.3}$$

The proof of this claim is the same as the proof of Lemma 4.2. We also notice that from the equation (4.17) we obtain the following relation:

$$\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i) \subset \boldsymbol{\alpha}_{P_{t_1}}(\bigcup_{i\notin I} v_i), \tag{5.4}$$

for $0 < t_1 < t_2 \leq 1$. In fact, from (4.17) it is straightforward to verify that

$$\bigcup_{0<t\leq 1} \boldsymbol{\alpha}_{P_t}(\bigcup_{i\notin I} v_i) = \bigcup_{i\notin I}(v_i)_{\frac{\pi}{2}}. \tag{5.5}$$

Examining equations (5.3), (5.4), and (5.5), we can separate three possible behaviors of $\rho_{P_t^*}(u)$ as a function of t for $t \leq 1$:

- If $u \in \boldsymbol{\alpha}_P(\bigcup_{i\notin I} v_i)$, then $u \in \boldsymbol{\alpha}_P(v_j)$ for some $j \notin I$. Thus, $r_P(v_j) \in H_P(u)$ which is equivalent to $r_{P^*}(u) \in H_{P^*}(v_j)$. Thus, from Lemma 4.2 we obtain

$$\rho_{P^*}(u) = \frac{\alpha_j}{uv_j}, \tag{5.6}$$

  where $j \notin I$, and, in particular, the minimum in (5.3) is attained at $j \notin I$. Therefore, from (5.3) we obtain that:

$$\rho_{P_t^*}(u) = t\rho_{P^*}(u). \tag{5.7}$$

- If $u \notin \bigcup_{i\notin I}(v_i)_{\frac{\pi}{2}}$, then $uv_i \leq 0$ for all $i \notin I$. Thus, from equation (5.3) (or equation (5.5)) we obtain that:

$$\rho_{P_t^*}(u) = \rho_{P^*}(u). \tag{5.8}$$

- If $u \in \bigcup_{i\notin I}(v_i)_{\frac{\pi}{2}} \setminus \boldsymbol{\alpha}_P(\bigcup_{i\notin I} v_i)$, then then by applying equations (5.3), (5.4), and (5.5) in a way similar to the previous two cases, we obtain that as t decreases, $\rho_{P_t^*}(u)$ remains constant up until the first moment when $u \in \boldsymbol{\alpha}_{P_t}(\bigcup_{i\notin I} v_i)$, after which it starts to scale. Let $t(u)$ be the maximum t for which $u \in \boldsymbol{\alpha}_{P_t}(\bigcup_{i\notin I} v_i)$. We obtain:

$$\begin{aligned} \rho_{P_t^*}(u) &= \rho_{P^*}(u) \text{ when } t \in [t(u), 1]. \\ \rho_{P_t^*}(u) &= \frac{t}{t(u)}\rho_{P^*}(u) \text{ when } t \in (0, t(u)]. \end{aligned} \tag{5.9}$$

By looking at the dual, using Lemma 4.4 and the convex hull equation (4.17) for P, we obtain the following behavior for $\rho_{P_t}(v_i)$:

- If $i \notin I$, then

$$\rho_{P_t}(v_i) = \frac{\rho_P(v_i)}{t}. \tag{5.10}$$

- If $i \in I$, then $\rho_{P_t}(v_i)$ is non-decreasing as t decreases and

$$\rho_{P_t}(v_i) \leq \frac{\rho_P(v_i)}{t}. \tag{5.11}$$

Now with the help of equations (5.7)-(5.11) regarding the behavior of ρ_{P_t} and $\rho_{P_t^*}$, we would like to compute $\Phi(P_{t_1}) - \Phi(P_{t_2})$ for some $0 < t_1 < t_2 \leq 1$. Using (5.4) and (5.5), we separate

$\Phi(P_{t_1}) - \Phi(P_{t_2})$ into the following terms:

$$\begin{aligned}
&\Phi(P_{t_1}) - \Phi(P_{t_2}) = \\
&\int \log(\rho_{P_{t_1}}) - \log(\rho_{P_{t_2}}) d\mu + \\
&\int_{\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)} \log(\rho_{P^*_{t_1}}) - \log(\rho_{P^*_{t_2}}) d\lambda + \\
&\int_{S^{n-1}\setminus\bigcup_{i\notin I}(v_i)_{\frac{\pi}{2}}} \log(\rho_{P^*_{t_1}}) - \log(\rho_{P^*_{t_2}}) d\lambda + \\
&\int_{\bigcup_{i\notin I}(v_i)_{\frac{\pi}{2}}\setminus\boldsymbol{\alpha}_{P_{t_1}}(\bigcup_{i\notin I} v_i)} \log(\rho_{P^*_{t_1}}) - \log(\rho_{P^*_{t_2}}) d\lambda \\
&\int_{\boldsymbol{\alpha}_{P_{t_1}}(\bigcup_{i\notin I} v_i)\setminus\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)} \log(\rho_{P^*_{t_1}}) - \log(\rho_{P^*_{t_2}}) d\lambda
\end{aligned} \tag{5.12}$$

Now, will estimate each term in the above equation.

The first term

First, we look at the integral with respect to μ. From (5.10) and since $\rho_{P_t}(v_i)$ is non-decreasing for $i \in I$, see (5.11), we obtain that:

$$\begin{aligned}
&\int \log(\rho_{P_{t_1}}) - \log(\rho_{P_{t_2}}) d\mu = \\
&\sum_{i\in I} \big(\log(\rho_{P_{t_1}}) - \log(\rho_{P_{t_2}})\big)\mu(v_i) + \sum_{i\notin I} \big(\log(\rho_{P_{t_1}}) - \log(\rho_{P_{t_2}})\big)\mu(v_i) \geq \\
&\sum_{i\notin I} \big(\log(\rho_{P_{t_1}}) - \log(\rho_{P_{t_2}})\big)\mu(v_i) \geq \\
&\log(\frac{t_2}{t_1})\mu(\bigcup_{i\notin I} v_i).
\end{aligned} \tag{5.13}$$

The second term

For the second term, if $u \in \boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)$, then $t_1 \leq t(u)$ and $t_2 \leq t(u)$. Thus, from (5.9) we obtain:

$$\begin{aligned}
&\log(\rho_{P^*_{t_2}}(u)) - \log(\rho_{P^*_{t_1}}(u)) = \\
&\log(\frac{t_1}{t(u)}\rho_{P^*}(u)) - \log(\frac{t_2}{t(u)}\rho_{P^*}(u)) = \\
&\log(\frac{t_1}{t_2}).
\end{aligned} \tag{5.14}$$

Therefore, for the second term, we obtain

$$\int_{\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)} \log(\rho_{P^*_{t_1}}) - \log(\rho_{P^*_{t_2}}) d\lambda = \log(\frac{t_1}{t_2})\lambda(\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)). \tag{5.15}$$

The third term
From (5.8), the third term is computed as::

$$\int_{S^{n-1}\setminus\bigcup_{i\notin I}(v_i)_{\frac{\pi}{2}}} \log(\rho_{P^*_{t_1}}) - \log(\rho_{P^*_{t_2}}) d\lambda = 0. \tag{5.16}$$

The fourth and the fifth terms
Now, we want to estimate the last two terms. As in (5.9) given

$$u \in \bigcup_{i\notin I}(v_i)_{\frac{\pi}{2}} \setminus \boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i) \tag{5.17}$$

let $t(u)$ denote the maximum t such that $u \in \boldsymbol{\alpha}_{P_t}(\bigcup_{i\notin I} v_i)$. Notice that if

$$u \in \bigcup_{i\notin I}(v_i)_{\frac{\pi}{2}} \setminus \boldsymbol{\alpha}_{P_{t_1}}(\bigcup_{i\notin I} v_i) \tag{5.18}$$

from (5.4) and (5.5) we obtain that $t(u) < t_1 < t_2$. And, thus, from (5.9) we obtain:

$$\log(\rho_{P^*_{t_1}}(u)) - \log(\rho_{P^*_{t_2}}(u)) = \log(\rho_{P^*}(u)) - \log(\rho_{P^*}(u)) = 0, \tag{5.19}$$

which implies that the fourth term is zero. On the other hand, if

$$u \in \boldsymbol{\alpha}_{P_{t_1}}(\bigcup_{i\notin I} v_i) \setminus \boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i), \tag{5.20}$$

then $t(u) \in [t_1, t_2)$. Thus, from (5.9) we obtain:

$$\begin{aligned} &|\int_{\boldsymbol{\alpha}_{P_{t_1}}(\bigcup_{i\notin I} v_i)\setminus\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)} \log(\rho_{P^*_{t_1}}) - \log(\rho_{P^*_{t_2}}) d\lambda| = \\ &|\int_{\boldsymbol{\alpha}_{P_{t_1}}(\bigcup_{i\notin I} v_i)\setminus\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)} \log(\frac{t_1}{t(u)}\rho_{P^*}(u)) - \log(\rho_{P^*_t}(u)) d\lambda| \leq \\ &\lambda\big(\boldsymbol{\alpha}_{P_{t_1}}(\bigcup_{i\notin I} v_i) \setminus \boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)\big)|\log(\frac{t_1}{t_2})|. \end{aligned} \tag{5.21}$$

From the continuity of measure λ and since λ is absolutely continious, as $t_1 \to t_2$ (with $t_1 < t_2$), we find that

$$\lambda\big(\boldsymbol{\alpha}_{P_{t_1}}(\bigcup_{i\notin I} v_i) \setminus \boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)\big) \to \lambda\big(\partial(\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i))\big) = 0, \tag{5.22}$$

where ∂ denotes the boundary of the set in S^{n-1}. So, in particular, given any $\varepsilon > 0$ for all t_1 sufficiently close enough to t_2, the right side at the end of (5.21) is less than $\varepsilon|\log(\frac{t_1}{t_2})|$.

Combining all of the above, we obtain that for $0 < t_1 < t_2 \leq 1$ such that t_1 is sufficiently close to t_2:

$$\begin{aligned} \Phi(P_{t_1}) - \Phi(P_{t_2}) &\geq \log(\frac{t_2}{t_1})\mu(\bigcup_{i\notin I} v_i) + \log(\frac{t_1}{t_2})\lambda(\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)) - \varepsilon|\log\frac{t_1}{t_2}| \\ &= \log(\frac{t_2}{t_1})(\mu(\bigcup_{i\notin I} v_i) - \lambda(\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)) - \varepsilon). \end{aligned} \tag{5.23}$$

Now, we are going to analyze two cases.

Case 1. Given $t_2 \in (0,1]$, *suppose* $\mu(\bigcup_{i\notin I} v_i) > \lambda(\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i))$.

Then (5.23), implies that there exists $t_1 < t_2$ such that for all $t \in [t_1, t_2)$, $\Phi(P_t) > \Phi(P_{t_2})$.

Case 2. Given $t_2 \in (0,1]$, *suppose* $\mu(\bigcup_{i\notin I} v_i) = \lambda(\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i))$.

Then, from (5.4) and (5.5),

$$\lambda\big(\boldsymbol{\alpha}_{P_{t_1}}(\bigcup_{i\notin I} v_i) \setminus \boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)\big) = 0. \tag{5.24}$$

This forces the estimate (5.21) to be equal to zero. Therefore, the equation (5.23) in this case refines to the following:

$$\begin{aligned} \Phi(P_{t_1}) - \Phi(P_{t_2}) &\geq \log(\frac{t_2}{t_1})\mu(\bigcup_{i\notin I} v_i) + \log(\frac{t_1}{t_2})\lambda(\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)) \\ &= \log(\frac{t_2}{t_1})(\mu(\bigcup_{i\notin I} v_i) - \lambda(\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i))) = 0. \end{aligned} \tag{5.25}$$

Therefore, in this case, for all $t \in (0, t_2]$ we obtain $\Phi(P_{t_1}) \geq \Phi(P_{t_2})$.

We are now prepared to conclude the proof. Recall that we were given $t_0 < 1$ such that

$$\mu(\bigcup_{i\notin I} v_i) \geq \lambda(\boldsymbol{\alpha}_{P_{t_0}}(\bigcup_{i\notin I} v_i)). \tag{5.26}$$

From (5.4), we obtain that the same statement holds for any $t_2 \in [t_0, 1]$:

$$\mu(\bigcup_{i\notin I} v_i) \geq \lambda(\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)). \tag{5.27}$$

Suppose now that $\Phi(P_{t_0}) < \Phi(P_1)$. Since $\Phi(P_t)$ is continuous, let $t_2 \in (t_0, 1]$ be the smallest value such that $\Phi(P_{t_2}) = \Phi(P_1)$. Then, for all $t \in [t_0, t_2)$, $\Phi(P_t) < \Phi(P_{t_2})$. If

$$\mu(\bigcup_{i\notin I} v_i) > \lambda(\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)) \tag{5.28}$$

then a contradiction arises from Case 1. If

$$\mu(\bigcup_{i\notin I} v_i) = \lambda(\boldsymbol{\alpha}_{P_{t_2}}(\bigcup_{i\notin I} v_i)) \tag{5.29}$$

Then, applying Case 2 would imply that $\Phi(P_{t_0}) = \Phi(P_{t_2})$, which is also a contradiction. Therefore, $\Phi(P_{t_0}) \geq \Phi(P_1)$, which was the desired. □

The next Lemma is a core component of the proof of Theorem 1.4, where we utilize two previous technical results: Lemma 4.3 and Lemma 5.1. Before we begin with the proof, let us first try to explain the statement of the next lemma in a more intuitive way. Suppose we are given some $P \in \mathcal{P}_\mu$, and a nonempty and not full indexing set $I \subset \{1 \dots m\}$. Suppose for this indexing set we have the following:

$$\begin{gathered} 0 < L^* \leq U^* < L \leq U = 1 \\ \frac{L}{U} \approx 1, \ \frac{U^*}{L} \approx 0, \ \frac{L^*}{U^*} \approx 1 \end{gathered} \tag{5.30}$$

Then, Lemma 5.2 claims that we can find a new polytope $P_r \in \mathcal{P}_\mu$ with $\Phi(P_r) \geq \Phi(P)$ such that:

$$\begin{gathered} 0 < L_r^* \leq U_r^* < L_r \leq U_r = 1 \\ \frac{L_r}{U_r} \approx 1,\ \frac{U_r^*}{L_r} \approx \cos(\frac{\pi}{2} - \alpha),\ \frac{L_r^*}{U_r^*} \approx 1, \end{gathered} \tag{5.31}$$

where α is the uniform weak Aleksandrov constant from Proposition 3.2.

Even though the Aleksandrov condition is not stated explicitly, it is embedded in the following Lemma as part of the assumption about:

$$\mu(\bigcup_{i \notin I} v_i) \leq \lambda(\bigcup_{i \notin I} (v_i)_{\frac{\pi}{2} - \alpha}). \tag{5.32}$$

We again recall that we use the notation established in (2.19)-(2.25).

Lemma 5.2. *Suppose $P \in \mathcal{P}_\mu$ is such that $\max_i \alpha_i = 1$, and $I \subset \{1 \dots m\}$ is a nonempty and not full indexing set. Suppose*

$$\mu(\bigcup_{i \notin I} v_i)) \leq \lambda(\bigcup_{i \notin I} (v_i)_{\frac{\pi}{2} - \alpha}) \tag{5.33}$$

for some $0 < \alpha < \frac{\pi}{2}$. Suppose $U^* < L\cos(\frac{\pi}{2} - \alpha)$. In particular, $0 < L^* \leq U^* < L \leq U = 1$.

We claim that there exists $P_r \in \mathcal{P}_\mu$, such that:

$$\begin{aligned} U_r &= 1 \\ L_r &= L \\ U_r^* &\leq L \cos(\frac{\pi}{2} - \alpha) \\ L_r^* &= \frac{L^*}{U^*} L \cos(\frac{\pi}{2} - \alpha) \end{aligned} \tag{5.34}$$

Moreover, $\alpha_i = \alpha_{r,i}$ for $i \in I$ and $\Phi(P_r) \geq \Phi(P)$.

Proof. We want to rescale P for indexing set I as:

$$P_t^* = \bigcap_{i \in I} H^-(t\alpha_i, v_i) \bigcap_{i \notin I} H^-(\alpha_i, v_i). \tag{5.35}$$

Note that this is opposite to the notation used in Lemma 5.1 and Lemma 4.4, where we rescaled indices $i \notin I$, but here we rescale indices $i \in I$. We define

$$t_0 = \frac{U^*}{L \cos(\frac{\pi}{2} - \alpha)} \tag{5.36}$$

From our assumptions, $0 < t_0 < 1$. Our goal is to analyze $P_{t_0}^*$ and to confirm that $\Phi(P_{t_0}) \geq \Phi(P)$ with the help of Lemma 5.1. Recall, the equation (5.3) from the proof of Lemma 5.1. Using it, we can write for $u \in S^{n-1}$:

$$\rho_{P_t^*}(u) = \min_{\forall i \text{ s.t.} uv_i > 0} \{\frac{t\alpha_i}{uv_i} \text{ if } i \in I \text{ or } \frac{\alpha_i}{uv_i} \text{ if } i \notin I\}. \tag{5.37}$$

From this, using definitions of L_t^*, U_t^*, L_t, U_t we immediately obtain from Lemma 4.4 the following:

$$\begin{aligned} U_t &= tU \\ L_t &= tL \\ U^* \geq U_t^* &\geq tU^* \end{aligned} \tag{5.38}$$

Notice that from the previous equation and the definition of t_0:

$$U_{t_0}^* \leq U^* = t_0 L \cos(\frac{\pi}{2} - \alpha) \leq L_{t_0} \cos(\frac{\pi}{2} - \alpha). \tag{5.39}$$

In particular,

$$U_{t_0}^* < L_{t_0}. \tag{5.40}$$

Therefore, we can apply Lemma 4.3 to P_{t_0} to obtain:

$$\lambda(\bigcup_{i \notin I} (v_i)_{\arccos \frac{U_{t_0}^*}{L_{t_0}}}) \leq \lambda(\boldsymbol{\alpha}_{P_{t_0}}(\bigcup_{i \notin I} v_i)). \tag{5.41}$$

This, combined with the assumption (5.33) and the estimate (5.39), gives us the following:

$$.\mu(\bigcup_{i \notin I} v_i)) \leq \lambda(\bigcup_{i \notin I} (v_i)_{\frac{\pi}{2} - \alpha}) \leq \lambda(\bigcup_{i \notin I} (v_i)_{\arccos \frac{U_{t_0}^*}{L_{t_0}}}) \leq \lambda(\boldsymbol{\alpha}_{P_{t_0}}(\bigcup_{i \notin I} v_i)). \tag{5.42}$$

Therefore, since $\boldsymbol{\alpha}_{P_{t_0}}(\bigcup_{i \notin I} v_i) \bigcap \boldsymbol{\alpha}_{P_{t_0}}(\bigcup_{i \in I} v_i)$ is λ measure zero, as it is Lebesgue measure zero, and since the measures have equal weights, we obtain from the above that the reverse holds for $\boldsymbol{\alpha}_{P_{t_0}}(\bigcup_{i \in I} v_i)$,

$$\mu(\bigcup_{i \in I} v_i) \geq \lambda(\boldsymbol{\alpha}_{P_{t_0}}(\bigcup_{i \in I} v_i)). \tag{5.43}$$

Therefore, we can apply Lemma 5.1 (again, notice the mentioned reverse of the notation for index set) to conclude:

$$\Phi(P_{t_0}) \geq \Phi(P). \tag{5.44}$$

We define P_r to be rP_{t_0} for some $r > 0$, so that $\max_i \alpha_{r,i} = 1$. We obtain,

$$\Phi(P_r) = \Phi(P_{t_0}) \geq \Phi(P). \tag{5.45}$$

What remains is to establish the values of U_r^*, L_r^*, L_r as well as $\alpha_{r,i}$ for $i \in I$.

Firstly, we notice from (2.5):

$$L_t^* \geq \min_{u \in S^{n-1}} h_{P_t^*}(u) = \min_{u \in S^{n-1}} \rho_{P_t^*}(u). \tag{5.46}$$

We are interested in whether $\rho_{P_t^*}(u)$ decreases. Notice that for $t \geq t_0$, we have

$$t \geq t_0 = \frac{U^*}{L \cos(\frac{\pi}{2} - \alpha)} \geq \frac{L^*}{L \cos(\frac{\pi}{2} - \alpha)} > \frac{L^*}{L}. \tag{5.47}$$

Now, from the previous equation, for any $i \in I$ if $uv_i > 0$, we have

$$\frac{t\alpha_i}{uv_i} > \frac{L^* \alpha_i}{L} > L^*. \tag{5.48}$$

We also have that for $i \notin I$ and $uv_i > 0$:

$$\frac{\alpha_i}{uv_i} \geq L^*. \tag{5.49}$$

Combining both previous equations and using (5.37), we obtain that $\rho_{P_t^*}(u) \geq L^*$ for any $u \in S^{n-1}$ and for any $t \geq t_0$. Therefore, we can apply (5.46) to deduce that $L_t^* \geq L^*$. This, combined with the fact that L_t^* can only decrease as t decreases and that $L_1^* = L^*$, imply that $L_t^* = L^*$ for $t \geq t_0$.

Summarizing, we obtain that for $t \geq t_0$:

$$\begin{aligned} U_t &= tU \\ L_t &= tL \\ U^* \geq & U_t^* \geq tU^* \\ L_t^* &= L^*. \end{aligned} \tag{5.50}$$

Recall that α_t is a representation for P_t. Notice that

$$\max_i \alpha_{t,i} = \max(U_t, U_t^*). \tag{5.51}$$

Now, since $U = 1$, from the definition of t_0 and (5.50) we obtain

$$U_t^* \leq U^* = t_0 L \cos(\frac{\pi}{2} - \alpha) < t_0 = t_0 U = U_{t_0}. \tag{5.52}$$

Therefore, combining two previous equations, we obtain

$$\max_i \alpha_{t,i} = \max(U_t, U_t^*) = U_t = t. \tag{5.53}$$

Recall that we defined P_r to be equal to rP_{t_0} such that $\max_i \alpha_{r,i} = 1$. In particular, from (5.47), we see that $r = t_0$. Thus, we obtain for P_r that:

$$\begin{aligned} &U_r = 1 \\ &L_r = \frac{L_{t_0}}{t_0} = L \\ &U_r^* = \frac{U_{t_0}^*}{t_0} \leq \frac{U^*}{t_0} = L \cos(\frac{\pi}{2} - \alpha) \\ &L_r^* = \frac{L_{t_0}^*}{t_0} = \frac{L^*}{U^*} L \cos(\frac{\pi}{2} - \alpha). \end{aligned} \tag{5.54}$$

Now, it only remains to show that for $i \in I$, we have that $\alpha_i = \alpha_{r,i}$. For this we apply equation $P_r = t_0 P_{t_0}$ and Lemma 4.4 for P_{t_0} (again noting that we rescale for $i \in I$ and not for $i \notin I$) to conclude that for $i \in I$,

$$\alpha_{r,i} = \frac{\alpha_{t_0,i}}{t_0} = \alpha_i. \tag{5.55}$$

□

6. Proof of the Main Result

We are ready to start the proof Theorem 1.4. Our strategy is first to pick a sequence of polytopes that maximize the functional Φ. Then, we will use Lemma 5.2 to modify this sequence, ensuring that it converges to a non-degenerate convex polytope. The proof heavily relies on the notations from (2.19)-(2.25) for varying index sets. We recall that, given a polytope with index $\mathfrak{n}$, as in $P_\mathfrak{n}$, and an index set $I \in \{1 \dots m\}$ we write

$$\begin{aligned} U_\mathfrak{n}(I) &:= \max_{i \in I} \alpha_i, \\ L_\mathfrak{n}(I) &:= \min_{i \in I} \alpha_i, \\ U^*_\mathfrak{n}(I) &:= \max_{i \notin I} \alpha_i, \\ L^*_\mathfrak{n}(I) &:= \min_{i \notin I} \alpha_i. \end{aligned} \tag{6.1}$$

where $\alpha_\mathfrak{n} = (\alpha_{\mathfrak{n},1}, \dots, \alpha_{\mathfrak{n},m})$ is the representation of $P_\mathfrak{n}$.

Proposition 6.1. *Suppose μ is a discrete measure not concentrated on a closed hemisphere and λ is an absolutely continuous Borel measure. Suppose μ is weakly Aleksandrov related to λ. Then there exists a sequence of polytopes $P_\mathfrak{n} \in \mathcal{P}_\mu$ maximizing $\Phi(\cdot)$, such that it converges to some $P \in \mathcal{P}_\mu$.*

Proof. Let $(P_\mathfrak{n})_{\mathfrak{n}=1}^\infty$ be any sequence that maximizes the functional. For each $\mathfrak{n}$, let $\alpha_\mathfrak{n}$ be the representation of $P_\mathfrak{n}$. Rescale each $P_\mathfrak{n}$ so that $\max_i \alpha_{\mathfrak{n},i} = 1$. Since the set $\{v_i \mid 1 \le i \le m\}$ is not contained in a closed hemisphere, it follows from $\max_i \alpha_{\mathfrak{n},i} = 1$ that there exists $R > 0$ such that for all $\mathfrak{n}$ we have $R_{P^*_\mathfrak{n}} < R$. Thus, we obtained a sequence that maximizes the functional and has a common bound on the outer radii of the duals.

For every permutation σ in S_m, where S_m represents the set of all possible permutations of m elements, we define the set $A_\sigma \subset \mathbb{N}$ to contain all indicies $\mathfrak{n}$ such that:

$$1 = \alpha_{\mathfrak{n},\sigma(1)} \ge \alpha_{\mathfrak{n},\sigma(2)} \ge \dots \alpha_{\mathfrak{n},\sigma(m)} > 0 \tag{6.2}$$

Since $\mathbb{N} = \cup_{\sigma \in S^{n-1}} A_\sigma$, there exists $\sigma \in S_m$ such that one of these sets is infinite. Without loss of generality, we let σ be the identity. We then take the subsequence of $(P_\mathfrak{n})_{\mathfrak{n}=1}^\infty$ containing only elements in A_σ. Since we will never use the original sequence, we redefine the constructed subsequence to be $(P_\mathfrak{n})_{\mathfrak{n}=1}^\infty$.

Thus, we obtain that for each $\mathfrak{n}$:

$$1 = \alpha_{\mathfrak{n},1} \ge \alpha_{\mathfrak{n},2} \ge \dots \ge \alpha_{\mathfrak{n},m} > 0 \tag{6.3}$$

Using standard compactness arguments, we can pass to the subsequence, which we again redefine to be $(P_\mathfrak{n})_{\mathfrak{n}=1}^\infty$, such that $P^*_\mathfrak{n}$ converges to some convex body K, which can be written as:

$$K = \bigcap_{i=1}^m H^-(\alpha_i, v_i), \tag{6.4}$$

where α_i are given by

$$\lim_{\mathfrak{n} \to \infty} \alpha_{\mathfrak{n},i} = \alpha_i. \tag{6.5}$$

Moreover, through repeated application of compactness, we can assume as well that for each $i \in \{1 \dots m-1\}$, there exists $\beta_i \in [0, 1]$:

$$\lim_{\mathfrak{n} \to \infty} \frac{\alpha_{\mathfrak{n},i+1}}{\alpha_{\mathfrak{n},i}} = \beta_i, \tag{6.6}$$

since $0 < \alpha_{\mathfrak{n},i+1} \leq \alpha_{\mathfrak{n},i}$ from (6.3). Notice that from (6.6) we also trivially obtain the following equation:

$$\lim_{\mathfrak{n}\to\infty} \frac{\alpha_{\mathfrak{n},i+k}}{\alpha_{\mathfrak{n},i}} = \prod_{0\leq j\leq k-1} \beta_{i+j}, \tag{6.7}$$

We also have the following equation for α_i:

$$1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_m \geq 0 \tag{6.8}$$

Note that if all coefficients $\alpha_i > 0$, then K contains the center in its interior, and, thus, $P_{\mathfrak{n}}$ will converge to $K^* = P \in \mathcal{P}_\mu$, which is what we desired. Suppose it is not the case. Then, in particular, from (6.8), we obtain that $\alpha_m = 0$. Since $\alpha_1 = 1$, we obtain that at least one of the variables β_i is equal to zero.

Now, we will construct indexing sets I_j, which correspond to what we call different rates of convergence of $\alpha_{\mathfrak{n},i}$ to zero. Suppose there are k indices for which β_i is equal to zero. Let $i_0 \leq i_2 \leq \ldots \leq i_{k-1}$ represent all indices such that $\beta_{i_j} = 0$. For convenience, let $i_k = m$. We define:

$$\begin{aligned} I_0 =& \{1, \ldots, i_0\} \\ I_1 =& \{i_0 + 1, \ldots, i_1\} \\ &\vdots \\ I_k =& \{i_{k-1} + 1, \ldots, m\} \end{aligned} \tag{6.9}$$

Notice that by construction, these sets are nonempty and not full, and their union is $\{1 \ldots m\}$. Moreover, from (6.3), (6.6), and from the definitions of sets I_j and indices i_j, we obtain the following inequalities:

$$1 = U_{\mathfrak{n}}(I_0) \geq L_{\mathfrak{n}}(I_0) \geq U_{\mathfrak{n}}(I_1) \geq L_{\mathfrak{n}}(I_1) > \ldots > U_{\mathfrak{n}}(I_k) \geq L_{\mathfrak{n}}(I_k) > 0 \tag{6.10}$$

$$\lim_{\mathfrak{n}\to\infty} \frac{U_{\mathfrak{n}}(I_{j+1})}{L_{\mathfrak{n}}(I_j)} = \lim_{\mathfrak{n}\to\infty} \frac{\alpha_{\mathfrak{n},i_j+1}}{\alpha_{\mathfrak{n},i_j}} = \beta_{i_j} = 0 \tag{6.11}$$

$$\lim_{\mathfrak{n}\to\infty} \frac{L_{\mathfrak{n}}(I_j)}{U_{\mathfrak{n}}(I_j)} = \lim_{\mathfrak{n}\to\infty} \frac{\alpha_{\mathfrak{n},i_j}}{\alpha_{\mathfrak{n},i_{j-1}+1}} > c_j \text{ for some constant } c_j > 0. \tag{6.12}$$

Intuitively, each I_j contains elements that converge to 0 at the same rate. For example, from (6.4), (6.10)-(6.12), we see that $i \in I_0$ if and only if $\alpha_i > 0$. On the other hand, elements with an index in the set I_K converge to zero the fastest. We call such a sequence of polytopes to be *degenerate of order* k. If we have a convergent sequence with limit K that is degenerate of order 0, then $K^* \in \mathcal{P}_\mu$. Therefore, to prove the proposition, it is sufficient to show that for a given sequence that is degenerate of order k, we can always find a new sequence of polytopes that is degenerate of a strictly lesser order than k and such that the new sequence still maximizes the functional. This will be our goal for the rest of the proof.

Let $I = I_0 \cup I_1 \ldots \cup I_{k-1}$. Note that the set $\{v_i \mid i \notin I\}$ is contained in a closed hemisphere as otherwise $P^*_{\mathfrak{n}}$ would converge to zero everywhere, which would contradict our assumption that $\max_i \alpha_{\mathfrak{n},i} = 1$. Since μ and λ are weak Aleksandrov related, using Proposition 3.2, we obtain that there exists a uniform weak Aleksandrov constant $\alpha > 0$. Therefore, since the set $\{v_i \mid i \notin I\}$ is a closed set and contained in a closed hemisphere, we can write:

$$\mu(\bigcup_{i\notin I} v_i)) \leq \lambda(\bigcup_{i\notin I}(v_i)_{\frac{\pi}{2}-\alpha}). \tag{6.13}$$

From (6.11) we can find N such that $\forall \mathfrak{n} > N$,

$$\frac{U_{\mathfrak{n}}^*(I)}{L_{\mathfrak{n}}(I)} = \frac{U_{\mathfrak{n}}(I_k)}{L_{\mathfrak{n}}(I_{k-1})} < \cos(\frac{\pi}{2} - \alpha). \tag{6.14}$$

It should also be noted that for all $\mathfrak{n}$ we have that $U_{\mathfrak{n}}(I) = U_{\mathfrak{n}}(I_0) = 1$. Thus, after combining this with (6.13) and (6.14), for each $\mathfrak{n} > N$, we can apply Lemma 5.2 to $P_{\mathfrak{n}}$ and the index set I to construct the partially rescaled polytopes $P_{r,\mathfrak{n}}$. Let $(P_{r,\mathfrak{n}})_{\mathfrak{n}=N}^{\infty}$ be the newly obtained sequence of partially rescaled polytopes. Let $\alpha_{r,\mathfrak{n}}$ be the representations of $P_{r,\mathfrak{n}}$.

First of all, notice that since $\Phi(P_{r,\mathfrak{n}}) \geq \Phi(P_{\mathfrak{n}})$, this sequence still maximizes the functional $\Phi(\cdot)$. Secondly, from Lemma 5.2, we also have that for $i \in I$,

$$\alpha_{r,\mathfrak{n},i} = \alpha_{\mathfrak{n},i}. \tag{6.15}$$

We also note the following identities based on Lemma 5.2:

$$\begin{aligned} U_{r,\mathfrak{n}}(I) =& 1 \\ L_{r,\mathfrak{n}}(I) =& L_{\mathfrak{n}}(I) \\ U_{r,\mathfrak{n}}^*(I) \leq& L_{\mathfrak{n}}(I)\cos(\frac{\pi}{2} - \alpha) \\ L_{r,\mathfrak{n}}^*(I) =& \frac{L_{\mathfrak{n}}^*(I)}{U_{\mathfrak{n}}^*(I)} L_{\mathfrak{n}}(I)\cos(\frac{\pi}{2} - \alpha) \end{aligned} \tag{6.16}$$

From (6.3), (6.15), and (6.16), we obtain:

$$1 = \alpha_{\mathfrak{n},1} = \alpha_{r,\mathfrak{n},1} \geq \alpha_{\mathfrak{n},2} = \alpha_{r,\mathfrak{n},2} \geq \ldots = \alpha_{r,\mathfrak{n},i_{k-1}} \geq \frac{U_{r,\mathfrak{n}}^*(I)}{\cos(\frac{\pi}{2} - \alpha)} > U_{r,\mathfrak{n}}^*(I) > 0 \tag{6.17}$$

Therefore, sets $I_0, I_1, \ldots, I_{k-2}$, which contained coefficients with a rate of convergence up to $k-2$, remain the same for the newly constructed sequence of polytopes $(P_{r,\mathfrak{n}})_{\mathfrak{n}=N}^{\infty}$. Moreover, I_{k-1} still consists of the coefficients that converge to zero with a rate $k-1$. We will show that for the subsequence of the newly constructed sequence of rescaled polytopes, the coefficients from the index set I_k converge with a rate $k-1$ as well.

From (6.16) and (6.12), for coefficients in I_k we have the following bound:

$$1 \geq \frac{L_{r,\mathfrak{n}}^*(I)}{U_{r,\mathfrak{n}}^*(I)} \geq \frac{L_{\mathfrak{n}}^*(I)}{U_{\mathfrak{n}}^*(I)} = \frac{L_{\mathfrak{n}}(I_k)}{U_{\mathfrak{n}}(I_k)} > c_k > 0 \tag{6.18}$$

Moreover, from (6.16) and (6.12), we have:

$$\frac{L_{r,\mathfrak{n}}^*(I)}{\alpha_{r,\mathfrak{n},i_{k-1}}} = \frac{L_{r,\mathfrak{n}}^*(I)}{L_{\mathfrak{n}}(I)} = \frac{L_{\mathfrak{n}}^*(I)}{U_{\mathfrak{n}}^*(I)}\cos(\frac{\pi}{2} - \alpha) > c_k\cos(\frac{\pi}{2} - \alpha) > 0 \tag{6.19}$$

While it is true that for $i \in I$, coefficients $\alpha_{r,\mathfrak{n},i}$ converge as they are equal to $\alpha_{\mathfrak{n},i}$, they might not do so for $i \notin I$. By applying compactness, we can ensure that they converge. Moreover, (6.18) guarantees that we can apply the same construction as in the beginning to ensure that all the ratios between the elements converge for $i \notin I$ to some values in the range $[c_k, \frac{1}{c_k}]$. Since the proof does not change, for the sake of notational simplicity, we assume that all elements, as well as all their ratios, converge for $i \notin I$ in $(P_{r,\mathfrak{n}})_{\mathfrak{n}=N}^{\infty}$. As at the beginning of the proof, we pick some subsequence, so that for some permutation σ of elements in I_K, we have the following based on (6.17):

$$1 = \alpha_{r,\mathfrak{n},1} \geq \alpha_{r,\mathfrak{n},2} \geq \ldots \geq \alpha_{r,\mathfrak{n},i_{k-1}} > U^*_{r,\mathfrak{n}}(I) = \alpha_{r,\mathfrak{n},\sigma(i_{k-1}+1)} \geq \ldots \geq \alpha_{r,\mathfrak{n},\sigma(m)} > 0 \tag{6.20}$$

This, combined with (6.19), establishes that all coefficients with index $i \in I_k$ converge the same as coefficient $\alpha_{r,\mathfrak{n},i_{k-1}}$:

$$\lim_{\mathfrak{n}\to\infty} \frac{\alpha_{r,\mathfrak{n},i}}{\alpha_{r,\mathfrak{n},i_{k-1}}} \geq \lim_{\mathfrak{n}\to\infty} \frac{L^*_{r,\mathfrak{n}}(I)}{\alpha_{r,\mathfrak{n},i_{k-1}}} > c_k \cos(\frac{\pi}{2} - \alpha) > 0. \tag{6.21}$$

Since $\alpha_{r,\mathfrak{n},i_{k-1}}$ converges with a rate $k-1$, we have constructed a sequence of polytopes that is degenerate of order $k-1$. This finishes the proof. □

The proof of Theorem 1.4 immediately follows by an application of Theorem 8.2 in [13].

Theorem 1.4 *Proof.* By Proposition 6.1, there exists $P \in \mathcal{P}_\mu \subset \mathcal{K}^n_o$ that maximizes the functional. Since μ is a Borel measure and λ is an absolutely continuous Borel measure, we infer from Theorem 8.2 in [13] that $\mu = \lambda(P, \cdot)$. □

To conclude, let us prove another Proposition which characterizes the bound on the inner to outer radius ratio for the solution to the Gauss Image Problem. The existence of the uniform constant comes from Proposition 3.2.

Proposition 6.2. *Suppose μ is a discrete measure that is not concentrated on a closed hemisphere and λ is an absolutely continuous Borel measure. Suppose μ is weak Aleksandrov related to λ. Let α be their uniform weak Aleksandrov constant. Then, there exists a polytope solution $P \in \mathcal{P}_\mu$ to the Gauss Image Problem such that the ratio $\frac{r_P}{R_p}$ is bounded from below by a constant depending only on vectors v_i and the uniform weak Aleksandrov constant α. Apart from α and vectors v_i, this constant is independent of λ.*

Proof. By Theorem 1.4, there exists $P \in \mathcal{P}_\mu$ solving the Gauss Image Problem for measures μ and λ. Consider any sequence of solutions $P_\mathfrak{n} \in \mathcal{P}_\mu$, with $\max_i \alpha_{\mathfrak{n},i} = 1$, that maximizes the ratio

$$\frac{\min_i \alpha_{\mathfrak{n},i}}{\max_i \alpha_{\mathfrak{n},i}}. \tag{6.22}$$

By compactness, there exists a subsequence converging to the body $P \in \mathcal{P}_\mu$. Since P still maximizes the functional, it is a solution by Theorem 8.2 in [13]. Let α be a representation for P. We also have that $\max_i \alpha_i = 1$ and if β is the representation for any other solution P' to the Gauss Image Problem, then

$$\frac{\min_i \alpha_i}{\max_i \alpha_i} \geq \frac{\min_i \beta_i}{\max_i \beta_i}. \tag{6.23}$$

For this P, reorder the index set so that $1 = \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_m > 0$. Define $I_l = \{1 \ldots l\}$. Let $k > 1$ be the integer such that the vectors $\{v_i \mid i \notin I_k\}$ are contained in a closed hemisphere, but the vectors $\{v_i \mid i \notin I_{k-1}\}$ are not. Clearly, $k < m-1$. Then, for any index set I_l with $l \geq k$, we have that $\{v_i \mid i \notin I_l\}$ are contained in a closed hemisphere. Thus, from Lemma 3.2, we have

$$\mu(\bigcup_{i\notin I_l} v_i)) \leq \lambda(\bigcup_{i\notin I_l} (v_i)_{\frac{\pi}{2}-\alpha}), \tag{6.24}$$

where α is the uniform weak Aleksandrov constant.

Suppose

$$U^*(I_l) < L(I_l)\cos(\frac{\pi}{2} - \alpha). \tag{6.25}$$

Then, we are able to apply Lemma 5.2 to find another body P_r such that

$$\begin{aligned} L_r(I_l) &= L(I_l) \\ L_r^*(I_l) &= \frac{L^*(I_l)}{U^*(I_l)} L(I_l)\cos(\frac{\pi}{2} - \alpha) > L^*(I_l) \end{aligned} \tag{6.26}$$

Notice that $L^*(I_l) < L(I_l)$. This, combined with (6.26), gives

$$\min\big(L_r(I_l)), L_r^*(I_l)\big) > L^*(I_l). \tag{6.27}$$

So, in particular, if α_r is the representation for P_r, then

$$\min_i \alpha_{r,i} = \min(L_r^*(I_l), L_r(I_l)) > L^*(I_l) = \min_i \alpha_i. \tag{6.28}$$

Therefore, we obtain that

$$\frac{\min_i \alpha_{r,i}}{\max_i \alpha_{r,i}} > \frac{\min_i \alpha_i}{\max_i \alpha_i}. \tag{6.29}$$

Since $\Phi(P_r) \geq \Phi(P)$, we have that P_r is still a solution to the Gauss Image Problem by Theorem 8.2 from [13]. Therefore, (6.29) contradicts (6.23).

Thus, we obtain that for all $l \geq k$ the opposite to (6.25) holds, that is

$$U^*(I_l) \geq L(I_l)\cos(\frac{\pi}{2} - \alpha). \tag{6.30}$$

In particular, since $1 = \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_m > 0$ we obtain for $l \geq k$

$$\alpha_{l+1} \geq \alpha_l \cos(\frac{\pi}{2} - \alpha). \tag{6.31}$$

Thus, we obtain

$$\alpha_m \geq \alpha_k \cos(\frac{\pi}{2} - \alpha)^{m-k}. \tag{6.32}$$

And, therefore,

$$\min_{u \in S^{n-1}} \rho_{P^*}(u) = \alpha_m \geq \alpha_k \cos(\frac{\pi}{2} - \alpha)^{m-k}. \tag{6.33}$$

Now, consider $\{v_i \mid i \notin I_{k-1}\}$. Define

$$\gamma = \inf_{u \in S^{n-1}, i \notin I_{k-1}} uv_i. \tag{6.34}$$

Since $\{v_i \mid i \notin I_{k-1}\}$ are not contained in a closed hemisphere, we obtain that $\gamma > 0$. Thus, from Lemma 4.2 we have that for all $u \in S^{n-1}$

$$\begin{aligned} &\rho_{P^*}(u) = \\ &\min_{\forall i \text{ s.t.} uv_i > 0} \frac{\alpha_i}{uv_i} \leq \\ &\min_{\forall i \notin I_{k-1} \text{ s.t.} uv_i > 0} \frac{\alpha_i}{uv_i} \leq \\ &\frac{\alpha_k}{\gamma}. \end{aligned} \tag{6.35}$$

Combining this with equation (6.33) we obtain:

$$\frac{r_P}{R_P} = \frac{r_{P^*}}{R_{P^*}} = \frac{\min \rho_{P^*}(u)}{\max \rho_{P^*}(u)} \geq \frac{\alpha_k \cos(\frac{\pi}{2} - \alpha)^{m-k}}{\frac{\alpha_k}{\gamma}} \geq \gamma \cos(\frac{\pi}{2} - \alpha)^{m-k}. \tag{6.36}$$

□

It would be interesting to consider whether the above approach can be used to solve the Gauss Image Problem with the weak Aleksandrov condition when λ is an absolutely continuous measure, and no additional discrete conditions are imposed on the measure μ, as was done in [13] for the classical Aleksandrov condition. The natural approach would be to discretize μ and to try to invoke Proposition 6.2. Yet, we notice that the bound on the inner to outer radius ratio for the solution to the discrete problem, obtained in the Proposition 6.2, significantly depends on the structure of this discretization.

Conjecture 6.3. *Suppose μ and λ are Borel measures on S^{n-1}, where λ is absolutely continuous. If μ is not concentrated on a closed hemisphere and is weakly Aleksandrov related to λ, then there exists a solution to the Gauss Image Problem.*

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