The Discrete Gauss Image Problem

Vadim Semenov

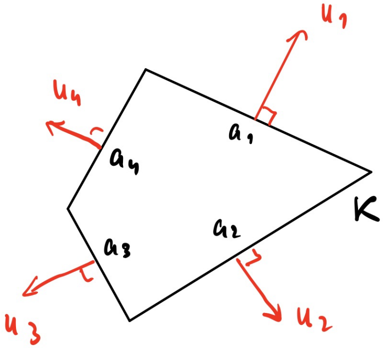
New York University, Courant Institute of Mathematical Sciences

Harmonic Analysis Methods in Geometric Tomography at ICERM
October 26, 2022

Introduction

Minkowski Problem (1903)

What are necessary and sufficient conditions for a Borel measure on the unit sphere to be the surface area measure of a convex body?



Fix $u_1, u_2, u_3, u_4 \in S^{n-1}$

Fix $a_1, a_2, a_3, a_4 > 0$

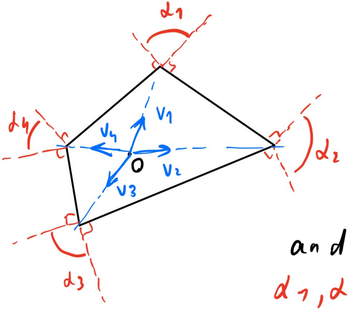
Find a Polytope K

s.t. Area of the facet in the direction u_i is equal to a_i.

What if we concentrate on angles at vertices instead of lengths of sides?

Alexandrov Problem (1939)

What are necessary and sufficient conditions for a Borel measure on the unit sphere to be the Alexandrov Integral curvature of a convex body?



Given v_1, v_2, v_3, v_4 and angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ find a Polytope K which has verticies at directions v_i and exterior angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$

Defenitions

$\mathcal{K}_o^n$ is the set of convex bodies with the center at their interior.
∂K is the boundary of K.
The radial map $r_K : S^{n-1} \to \partial K$ is defined by

$$r_K(u) = ru \in \partial K. \tag{1}$$

By $N(K,x)$, we denote the *normal cone of K at* $x \in \partial K$, that is the set of all outer unit normals at x:

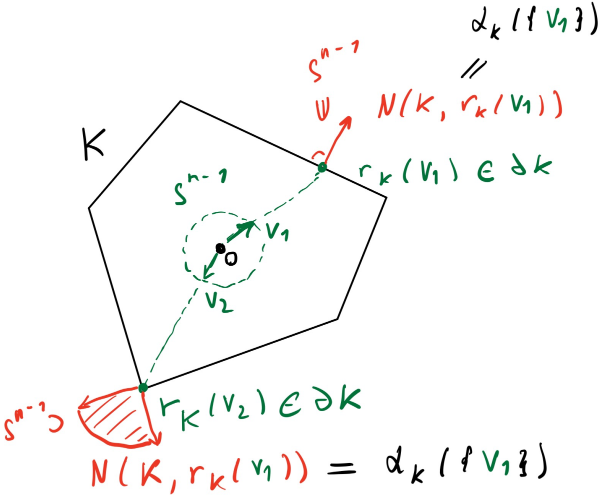
$$N(K,x) = \{v \in S^{n-1} : (y - x) \cdot v \leq 0 \text{ for all } y \in K\}. \tag{2}$$

We define the radial Gauss image of $\omega \subset S^{n-1}$ as:

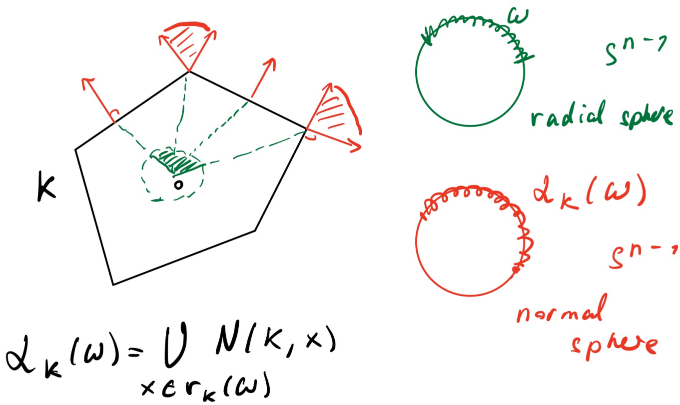
$$\boldsymbol{\alpha}_K(\omega) = \bigcup_{x \in r_K(\omega)} N(K,x) \subset S^{n-1}. \tag{3}$$

The radial Gauss image $\boldsymbol{\alpha}_K$ maps sets of S^{n-1} into sets of S^{n-1}.

Definitions:



Definition of the radial Gauss image map $\boldsymbol{\alpha}_K$ mapping sets of S^{n-1} to sets of S^{n-1}. The radial Gauss image map is a composition of Gauss map with radial map.

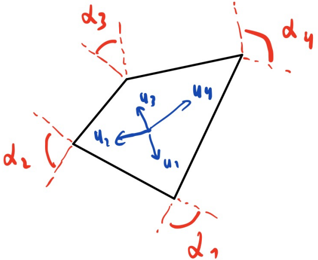


Definition

Alexandrov Integral curvature of a convex body $K \in \mathcal{K}_o^n$ is a pullback of the Lebesgue measure on S^{n-1} under $\boldsymbol{\alpha}_K$ map. That is, for each Borel $\omega \subset S^{n-1}$ we define a measure $\lambda(K, \cdot)$ where λ is Lebesgue measure on the sphere and

$$\lambda(\boldsymbol{\alpha}_K(\omega)) = \lambda(K, \omega). \tag{4}$$

Alexandrov integral curvature measures "the amount of normals" in a given radial directions. For polytopes in dimension two this measure is just angles at verticies.



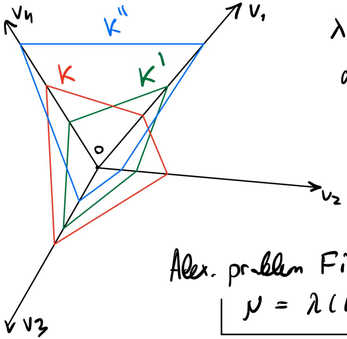
$$\lambda(K, \cdot) = \sum_{i=1}^{4} \alpha_i \delta_{u_i}$$

$\delta_{u_i}(v) = 1$ iff $v = u_i$

$= 0$ otherwise

Suppsoe $\mu = \sum_{i=1}^{m} \mu_i \delta_{v_i}$. Then, trying to solve the Alexandrov problem ($\mu = \lambda(K, \cdot)$) we can assume that K is a polytope with verticies $r_P(v_i)$.

While the Discrete Minkowski Problem fixes directions of facets and their area, the Discrete Alexandrov Problem fixes directions of vertices and their "spherical area." In some sense, this is a dual problem.

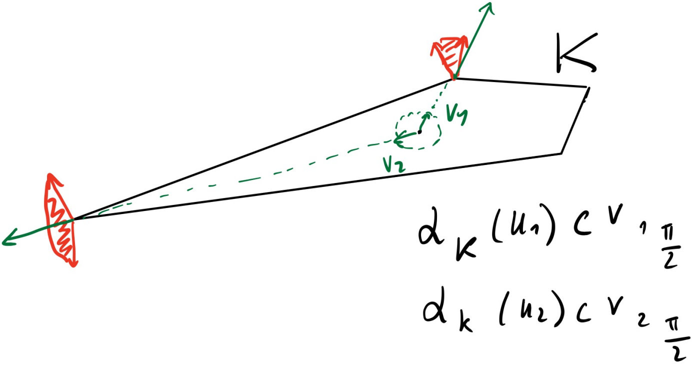


$\lambda(K,\cdot)$, $\lambda(K',\cdot)$ $\lambda(K'',\cdot)$ are discrete measures with weights at v_i.

Alex. problem Find K s.t. $\mu = \lambda(K,\cdot)$

Necessary conditions

What can $\lambda(K,\cdot)$ be? Notice that
$\boldsymbol{\alpha}_K(v) \subset v_{\frac{\pi}{2}} := \{u \in S^{n-1} \mid u \cdot v > \cos\frac{\pi}{2} = 0\}$. Hence,
$\boldsymbol{\alpha}_K(\omega) \subset \omega_{\frac{\pi}{2}} := \bigcup_{v\in\omega}\{u \in S^{n-1} : u \cdot v > 0\}$.

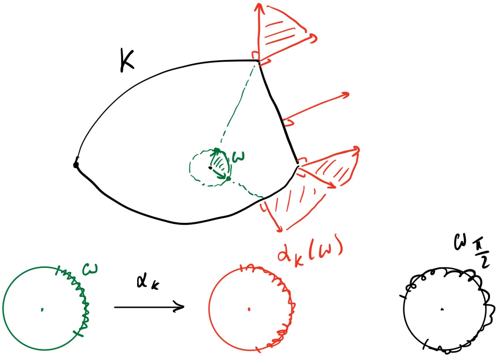


Notice that Aleksandrov's integral Curvature for any body K satisfies the following:

$$\lambda(S^{n-1}) = \lambda(K, S^{n-1}). \tag{5}$$

$$\lambda(K, \omega) < \lambda(\omega_{\frac{\pi}{2}}). \tag{6}$$

where second inequality follows by simple set containment. (To be fair, $\lambda(K, \omega) \leq \lambda(\omega_{\arccos \frac{r_K}{R_K}}) < \lambda(\omega_{\frac{\pi}{2}})$)



This exactly provides a necessary condition: [1]

Definition

Two Borel measures μ and λ (not necessarily Lebesgue measure) on S^{n-1} are called Alexandrov related if $\mu(S^{n-1}) = \lambda(S^{n-1})$, and for each compact spherically convex set $\omega \subset S^{n-1}$,

$$\mu(\omega) < \lambda(\omega_{\frac{\pi}{2}}). \tag{7}$$

It turns out to be sufficient:

Theorem (Alexandrov)

Suppose μ is a Borel measures on S^{n-1}. If μ and Lebesgue measure λ are Alexandrov related, then there exists a body $K \in \mathcal{K}_o^n$ such that $\mu = \lambda(K, \cdot) := \lambda(\boldsymbol{\alpha}_K(\cdot))$.

[1] The original definition in the Alexandrov paper is different but equivalent. He only considered λ to be Lebesgue measure. This definition is also a bit different from the 2019 GIP paper (BLYZZ).

Up to this point, λ was always the Lebesgue measure on S^{n-1}. What happens if we allow different measures instead of λ?

Definition (K. J. Böröczky, E. Lutwak, D. Yang, G. Y. Zhang and Y. M. Zhao, 2019)

The Gauss image measure of λ *via* K, is a measure defined as the pushforward of the λ via map $\boldsymbol{\alpha}_K$. That is for each borel $\omega \subset S^{n-1}$

$$\lambda(\boldsymbol{\alpha}_K(\omega)) = \lambda(K, \omega) \tag{8}$$

1. λ is spherical Lebesgue measure $\implies$ $\lambda(K, \cdot)$ is Alexandrov's integral curvature
2. λ is Federer's $(n-1)^{\text{th}}$ curvature measure $\implies$ $\lambda(K, \cdot)$ is the surface area measure of Alexandrov-Fenchel-Jessen
3. Dual curvature measures (the dual counterparts of Federer's curvature measures) are also Gauss Image Measures

The Gauss Image problem (K. J. Böröczky, E. Lutwak, D. Yang, G. Y. Zhang and Y. M. Zhao, 2019)

What are necessary and sufficient conditions for a Borel measure μ on the unit sphere to be the Gauss image measure of λ via some $K \in \mathcal{K}_o^n$?

Theorem (K. J. Böröczky, E. Lutwak, D. Yang, G. Y. Zhang and Y. M. Zhao, 2019)

Suppose μ and λ are Borel measures on S^{n-1} and λ is absolutely continuous. If μ and λ are Alexandrov related, then there exists a $K \in \mathcal{K}_o^n$ such that $\mu = \lambda(K, \cdot)$.

If λ is absolutely continuous and strictly positive on open sets

- $\implies$ Alexandrov relation is a necessary assumption.
- $\implies$ Solution is unique up to a dilation.

These are different from discrete.

Proposition (K. J. Böröczky, E. Lutwak, D. Yang, G. Y. Zhang and Y. M. Zhao, 2019)

If λ is an absolutely continuous measure on S^{n-1} then $\lambda(K, \cdot)$ is a valuation.

It is a variational proof. Define:

$$\Phi(K, \mu, \lambda) := \int \log \rho_K d\mu + \int \log \rho_{K^*} d\lambda. \tag{9}$$

Then the main steps are:

- Show that if $K \in \mathcal{K}_o^n$ maximizes $\Phi(\cdot, \mu, \lambda)$ then it solves $\mu = \lambda(K, \cdot)$.
- Obtain from the Alexandrov relation a "stronger" version with uniform constants δ and α:

$$\mu(\omega) < (1 - \delta)\lambda(\omega_{\frac{\pi}{2} - \alpha}) \tag{10}$$

- Using the stronger version, show that if for a given sequence of bodies K_i, if $\frac{r_{K_i}}{R_{K_i}} \to 0$ then $\Phi(K_i, \mu, \lambda) \to -\infty$. (Or equivalently show that if K is a very thin body then it is far from maximizing the functional.)
- Complete the proof with compactness argument.

Which parts rely on absolutely continuity? All of the above.

The Discrete Gauss Image Problem

Given discrete measures μ and λ:

$$\lambda = \sum_{j=1}^{k} \lambda_j \delta_{u_j} \qquad \mu = \sum_{i=1}^{m} \mu_i \delta_{v_i}. \tag{11}$$

The Discrete Gauss Image problem

Does there exist a body $K \in \mathcal{K}_o^n$ such that $\mu = \lambda(K, \cdot)$.

Proposition (S.)

If $\mu = \lambda(K, \cdot)$ for some $K \in \mathcal{K}_o^n$, then a convex polytope P with vertices at $r_K(v_i)$ also satisfies $\mu = \lambda(P, \cdot)$

Thus, we might restrict our attention to polytopes P_μ. Where $P \in P_\mu$ if and only if P is a convex polytope with vertices $r_P(v_i)$.

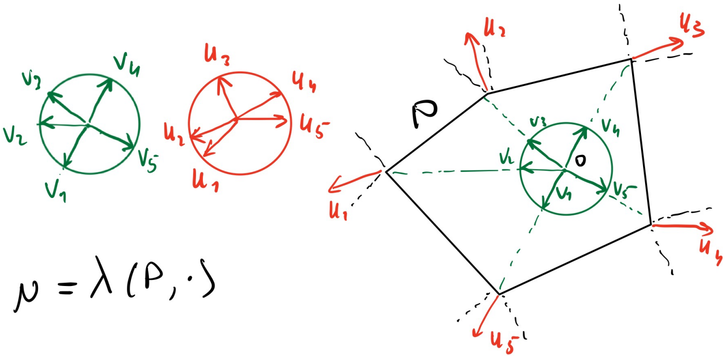
For this talk

While there is a machinery and technique developed to deal with all possible discrete measures and the statements hold in much more generality for the purpose of this talk we are going to restrict the class to the most simple discrete measures. Equal-weight μ and λ.

From now on, measures μ and λ are assumed to be the following $\lambda = \sum_{j=1}^{m} \delta_{u_j}$ and $\mu = \sum_{i=1}^{m} \delta_{v_i}$. The problem has a very simple geometric statement:

Discrete Equal-Weight Problem (S.)

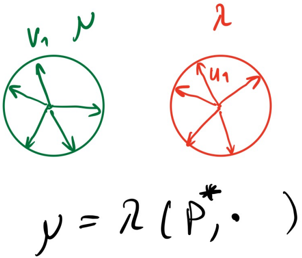
Suppose we are given two sets of unit vectors $\{v_1 \dots v_m\}$ and $\{u_1 \dots u_m\}$. Suppose $\{v_1 \dots v_m\} \subset S^{n-1}$ are not contained in any closed hemisphere. What are necessary and sufficient conditions on vectors v_i, u_j for the existence of a convex polytope P with vertices $r_P(v_i)$, such that every normal cone at each vertex of P contains exactly one vector from the set $\{u_1 \dots u_m\}$ in its interior?

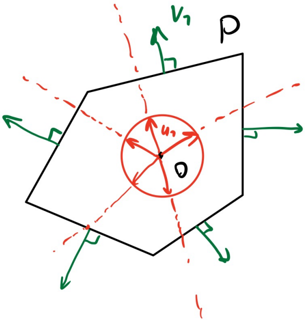


Or alternatively:

Discrete Equal-Weight Problem (S.)

What are necessary and sufficient conditions on vectors v_i, u_j for the existence of a convex polytope P with facets in the directions v_i, such that each facet is penetrated in its interior by exactly one ray $\{ut \mid t \geq 0\}$ where $u \in \{u_1 \dots u_m\}$?





Definition (S.)

Given a discrete measure λ and a discrete measure μ, we associate *the set of assignment functions*:

$$\mathbb{F}_{\mu,\lambda} := \left\{ f\colon \{1\ldots k\} \to \{1\ldots m\} \mid \sum_{j\in f^{-1}(i)} \lambda_j = \mu_i \right\}. \tag{12}$$

In this talk case $\mathbb{F}_{\mu,\lambda}$ can be thought of as a set of permutations on the set of $m = k$ elements. Each function gives an assignment of normal vector to normal cone at a vertex.

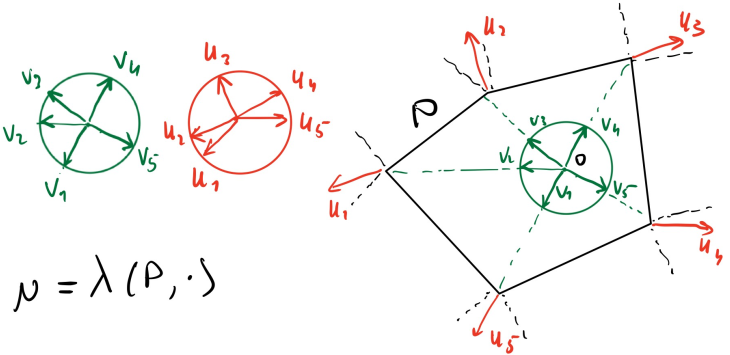
Question

- Does the solution Polytope exist for every assignment function (permutation)?

This is a proper reformulation of uniqueness question.

Preparation for the main results

On the interior assumption of $u_j \in \boldsymbol{\alpha}_P(\mathring{v}_{\sigma(i)})$.



Turns out Alexandrov condition is not quite right for discrete measures:

- $\mu(\omega) < \lambda(\omega_{\frac{\pi}{2}})$. Classical Alexandrov Condition. Equivalent to $\mu(\omega) + \lambda(\omega^*) < \mu(S^{n-1}) = \lambda(S^{n-1})$.
- $\mu(\omega) < (1-\delta)\lambda(\omega_{\frac{\pi}{2}-\alpha})$. GIP 2019 paper
- $\mu(\omega) \leq \lambda(\omega_{\frac{\pi}{2}-\alpha})$ where $\alpha > 0$. (Uniform) What actually happens if $\mu = \lambda(K, \cdot)$, where $K \in \mathcal{K}_o^n$. (Proof sketch: For any vector v, $\boldsymbol{\alpha}_K(v) \subset v_{\arccos \frac{r_K}{R_K}}$)
- $\mu(\omega) \leq \lambda(\omega_{\frac{\pi}{2}})$;

where

$$\omega_{\frac{\pi}{2}-\alpha} = \bigcup_{u \in \omega} \{v \in S^{n-1} : u \cdot v > \cos(\frac{\pi}{2} - \alpha)\}. \tag{13}$$

There is also a choice between ω being a spherically convex compact set or any compact set (or some other classes of sets, and how they relate to each other).

Measures	$\mu(\omega) < \lambda(\omega_{\frac{\pi}{2}})$	$\mu(\omega) \leq \lambda(\omega_{\frac{\pi}{2}})$
λ is Absolutely Continious	Good	Wrong
Both are Discrete	Wrong	Good

Definition (S.)

Two discrete measures μ and λ are called weak Alexandrov related if for each compact spherically convex set $\omega \subset S^{n-1}$, $\mu(\omega) \leq \lambda(\omega_{\frac{\pi}{2}})$.

Proposition (S.)

Suppose discrete μ and λ are weak Alexandrov related. Then there exist a uniform $\alpha > 0$ such that for each closed set $\omega \subset S^{n-1}$, $\mu(\omega) \leq \lambda(\omega_{\frac{\pi}{2}-\alpha})$.

This recovers the "true" condition.

Recall that $\alpha_K(v) \subset v_{\frac{\pi}{2}} := \{u \in S^{n-1} \mid u \cdot v > \cos\frac{\pi}{2} = 0\}$.

Proposition (S.)

Weak Alexandrov relation is a necessary assumption.

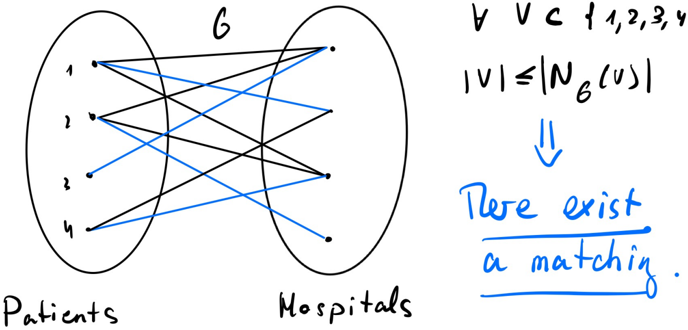
Before we even attempt to solve the problem we need to ask the following question.

Question

Given the weak Alexandrov condition, does there exist an assignment function $\sigma \in \mathbb{F}_{\mu,\lambda}$ such that $\forall j\ u_j v_{\sigma(j)} > 0$?

If μ and λ are discrete equal-weight then the weak Alexandrov condition is equivalent to Hall's Marriage Theorem.

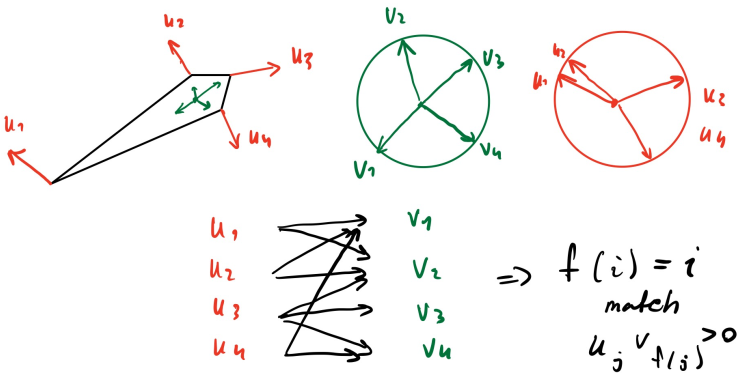
Let G be a finite bipartite graph with bipartite sets of verticies X (patients) and Y (hospitals).



Proposition (S.)

Hall's marriage condition is equivalent to the weak Aleksandrov condition. In particular, if one of the conditions is satisfied, there exist an $\sigma \in \mathbb{F}_{\mu,\lambda}$ such that $\forall j\ u_j v_{\sigma(j)} > 0$

Making all Alexandrov conditions vary natural as they are requirements from set containment.



Definition (S.)

For each $\sigma \in \mathbb{F}_{\mu,\lambda}$ we define the assignment functional

$$A(\sigma) := \sum_{j=1}^{k} \log u_j v_{\sigma(j)}, \tag{14}$$

where the log of negative values is forced to be $A(\sigma) = -\infty$. If there exist a solution to the Gauss Image Problem, we call its assignment function a solution function.

Main results

Theorem (Existence and Uniqueness (S.))

Let λ be a discrete equal-weight measure and μ be a discrete measure. Suppose they are weak Aleksandrov related and μ is not concentrated on a closed hemisphere. Then, $f \in \mathbb{F}_{\mu,\lambda}$ is a solution function if and only if it is the unique maximizer of the assignment functional. In other words,

- *The assignment functional, $A(f)$, is maximized at exactly one $f \in \mathbb{F}$. For this f, there exists a polytope $P \in \mathcal{P}_\mu$ such that $\lambda(P, \cdot) = \mu$ and $u_j \in \boldsymbol{\alpha}_P(\mathring{v}_{f(j)})$.*
- *Or $A(f)$ is maximized at more than one $f \in \mathbb{F}$, in which case there is no convex body $K \in \mathcal{K}^n_o$ such that $\lambda(K, \cdot) = \mu$.*

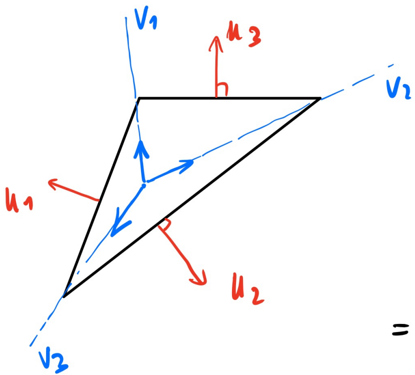
$$A(f) := \sum_{j=1}^{k} \log u_j v_{f(j)}, \tag{15}$$

It can be seen that generically (among measures μ and λ satisfying weak Alexandrov relation) maximizer is unique.[2]

Geometrically, when is it true that the assignment functional is uniquely maximized?

[2] almost everywhere, dense open in regular topology or Zariski topology

In dimension 2, all counterexamples come from polytopes.



$$\lambda = \sum_{i=1}^{3} \lambda_i \delta_{v_i}$$

$$\mu = \sum_{i=1}^{3} \mu_i \delta_{u_i}$$

$$A(Id) = A(i \overset{\text{perm}}{\to} i+1)$$

$$= \max_{\sigma} A(\sigma)$$

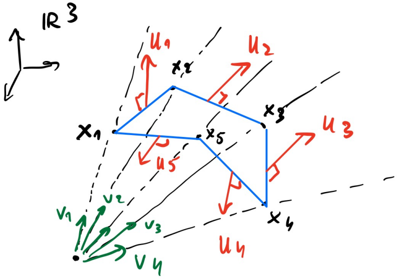
Thus $\mu = \lambda(K, \cdot)$ doesn't have a solution.

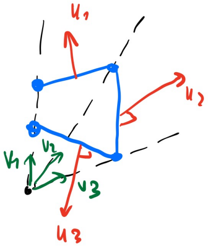
Definition (S.)

Given two sets of vectors $\{u_1 \dots u_l\}$ and $\{v_1 \dots v_l\}$ suppose there exist a piecewise linear closed curve with vertices $\{x_1 \dots x_l\}$ such that

- $x_i = \lambda_i v_i$ for some $\lambda_i > 0$
- $u_i \perp [x_i, x_{i+1}]$ for $1 \leq i \leq l-1$
- $u_l \perp [x_l, x_1]$

This curve is called an edge normal loop of two sets.





Definition (S.)

We call two sets $\{u_1 \dots u_k\}$ and $\{v_1 \dots v_k\}$ are to be edge loop free, if for any $\sigma, \sigma' \in S_k$ and given any l such that $2 \leq l \leq k$, there dose not exist and edge normal loop for $\{u_\sigma(1) \dots u_{\sigma(l)}\}$ and $\{v_{\sigma'(1)} \dots v_{\sigma'(l)}\}$.

Proposition (S.)

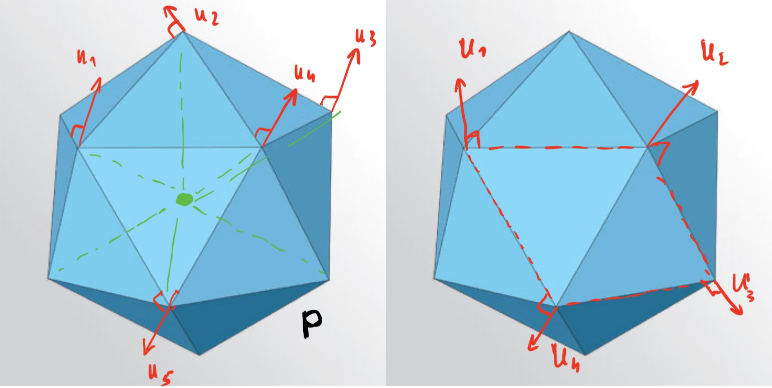
Suppose μ and λ are two equal-weight discrete measures such that $\{u_1 \dots u_k\}$ and $\{v_1 \dots v_k\}$ are loop free, then for any $\sigma_1, \sigma_2 \in \mathbb{F}_{\mu,\lambda,p}$

- *$\sigma_1 = \sigma_2$ if and only if $A(\sigma_1) = A(\sigma_2)$.*

Corollary (S.)

Suppose μ and λ are two equal-weight discrete measures. Suppose μ and λ are weak Alexandrov related and that μ is not concentrated on closed hemisphere. Suppose that $\{u_1 \dots u_k\}$ and $\{v_1 \dots v_k\}$ are loop free. Then there exist $K \in \mathcal{K}_o^n$ such that $\mu = \lambda(K, \cdot)$.

There is another way to think about it. Suppose you start with some convex polytope with vertices $r_P(v_i)$. Suppose $u_j \subset \boldsymbol{\alpha}_P(v_j)$. Can you change the vertices along the rays v_i, so that $u_j \subset \mathring{\boldsymbol{\alpha}}_P(v_j)$? If μ and λ are loop free you can.



$r_P(v_5) \nearrow,\ r_P(v_1) \nearrow,\ r_P(u_2) \nearrow\nearrow,\ r_P(u_4) \nearrow\nearrow\nearrow,\ r_P(u_5) \nearrow\nearrow\nearrow$

Sketches of the Proofs

There are two proofs of the main results. Variational proof (stochastic matrices, Transportation polytopes and Birkhoff-von-Neumann theorem) and a proof based on Helly's theorem and systems of equations.

Recall, $\Phi(K, \mu, \lambda) := \int \log \rho_K d\mu + \int \log \rho_{K^*} d\lambda$.
Variational proof skatch:

1. For any $f \in \mathbb{F}_{\mu,\lambda}$ we have $\Phi(K, \mu, \lambda) \leq -A(f)$.
2. Smooth out measure λ to λ_ε. (For small enough smoothing we have $\mu(\omega) < \lambda_\varepsilon(\omega_{\frac{\pi}{2}-\alpha})$ where α is uniform on ε and ω)
3. Prove the Gauss Image Paper under weak Alexandrov condition for μ discrete and λ absolutely continious. (S. 2022 other work) One needs to find a solution P such that $\frac{r_P}{R_p}$ is bounded from below by a constant, depending only on vectors v_i and the uniform weak Aleksandrov constant α. Besides being dependent on α, this constant is independent of λ.
4. Consider sequence K_ε of solutions to μ, λ_ε-problem and using bounds from α etc. show that there exist $K_\varepsilon \to P$.
5. Prove that $\Phi(K_\varepsilon, \mu, \lambda_\varepsilon) \to \Phi(K, \mu, \lambda)$.

- This gives (after some geometric computation):

$$\Phi(K,\mu,\lambda) = -\sum_{i,j=1}^{i=m,j=m} c_{i,j}\log(v_i u_j) > 0. \quad (16)$$

where $c_{i,j}$ is doubly stochastic matrix.

Using Birkhoff-von-Neumann theorem or results on transportation polytopes obtain that here exist $0 \le \theta_f \le 1$ for $f \in \mathbb{F}$, such that $\sum_{f\in\mathbb{F}} \theta_f = 1$ and

$$\Phi(K,\mu,\lambda) = -\sum_{f\in\mathbb{F}} \theta_f A(f). \quad (17)$$

Combining this with the first result, that for any $f \in \mathbb{F}_{\mu,\lambda}$ we have $\Phi(K,\mu,\lambda) \le -A(f)$, we obtain that we only have a sum of maximizers in $\Phi(K,\mu,\lambda)$. Arrive to the conclusion abouts the behavior of $\Phi(K_\varepsilon,\mu,\lambda_\varepsilon)$ and its convergence to obtain the main theorem.

Alternative proof of the main result for equal-weight measures:

1. Show that solution exists if and only if the following system of equations is solvable for $a_{i,j}$ where $i \neq j$:

$$a_{j,i} < x_j - x_i \text{ for } i \neq j \in \{1 \ldots m\} \tag{18}$$

   where $a_{j,i} = \log \frac{u_j v_i}{u_j v_j}$ if $\frac{u_j v_i}{u_j v_j} > 0$, and $a_{i,j} = -\infty$ otherwise.[3]

2. Show that the identity permutation is the unique maximizer if and only if for any non-trivial permutation σ on $\{1 \ldots m\}$,

$$a_\sigma := \sum_{i=1}^{m} a_{i,\sigma(i)} < 0 \tag{19}$$

3. Using, Helly's Theorem [4] show that the system is solvable if and only if for any non-trivial permutation σ on $\{1 \ldots m\}$,

$$a_\sigma := \sum_{i=1}^{m} a_{i,\sigma(i)} < 0 \tag{20}$$

[3] We define $a_{i,i} = 0$ for convenience.

[4] Similar system can be seen in Kasia Wyczesany's works

Thank you!